THE

MATHEMATICAL GAZETTE

EDITED BY
T. A. A. BROADBENT, M.A.

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Intending members are requested to communicate with one of the Secretaries (G. L. Parsons, Peckwater, Eastcote Road, Pinner, Middlesex; Miss Punnett, 17 Gower St., London, W.C.1). The subscription to the Association is 15s. per annum, and is due on Jan. 1st. It includes the subscription to "The Mathematical Gazette".

Change of Address should be notified to Miss Punnett. If Copies of the "Gazette" fail for lack of such notification to reach a member, duplicate copies can be supplied only at the published price.

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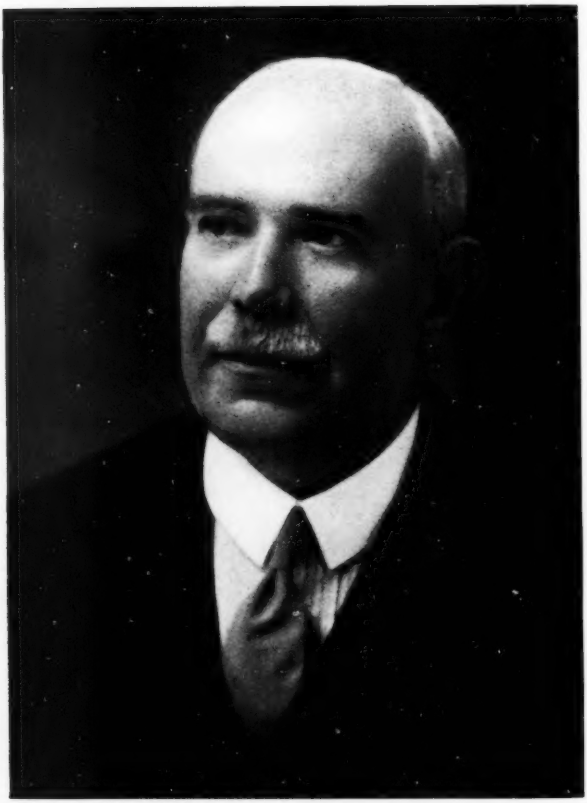

LOUIS NAPOLEON GEORGE FILON,
C.B.E., T.D., D.Sc., F.R.S.
President, January 1937 to December 1937.

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VOL. XXII. FEBRUARY, 1938. No. 248

Professor L. N. G. FILON, C.B.E., M.A., D.Sc., F.R.S. 1875-1937.

THE Association will mourn the untimely death of its President, Professor Louis Napoleon George Filon, less than a week before the Annual Meeting to which he had looked forward with such keen anticipation and at which he had hoped to preside and to give his Presidential Address. It is easy for those who knew him and his work to understand how strongly the objects of the Association appealed to him and how deeply he appreciated his election as its President.

He was of French descent, educated and trained as a mathematician in England. In his life and work he presented a peculiarly happy combination of much that is great in the characteristic traditions of the two countries in mathematical thought. He was intensely interested in the systematic and logical presentation of mechanics. His Presidential Address is the outcome of many years of thought and study devoted to this topic which was of all topics perhaps the nearest to his heart. The views he there sets out were modified and developed in the course of his work as a practical teacher of students. One great loss by his death, to many of us second only to the loss of a dear friend and a good colleague, is that his hope and intention to write a comprehensive book on the subject can now not be realised.

At the same time he had a very practical and experimental approach to his subject. His mind worked very much after the manner of Newton whose devout disciple he was. To him mechanics was an experimental science and he was impatient with those who would use it merely as an opportunity for pure mathematical manipulation. After he had lectured to his students on Lagrange's equations and the theory of oscillations he sent them into a laboratory where they could handle and experiment with tops and gyrostats and observe and measure their precessions.

The same practical bent showed itself in other directions. Earlier in life as a teacher of pure mathematics he did a great deal by his teaching and writing to give pure geometry its rightful place in university studies. How many of us who know that a conic can be drawn through any five points have ever marked five points at ran-

dom on a sheet of paper and actually drawn the conic which passes through them? Filon firmly believed that every student of geometry should have this experience and others like it at least once in his life. Accordingly side by side with his lectures on projective geometry there were practical classes in which his students with drawing-board and compasses made an experimental exploration which gave them a more vivid appreciation of geometrical truth than they otherwise would have had.

Under the same impulse he transformed the teaching of astronomy in his College. He found it largely theoretical in its treatment, though under his predecessor and revered teacher Karl Pearson practical instruction had been given in a small observatory in the College grounds. He left it as a highly developed subject in many of its branches and as Director of an active University Observatory at Mill Hill.

All his old students will remember his lectures. Prepared with a characteristic thoroughness, they were delivered in a clear and unhesitating manner and illustrated by a supreme command of the blackboard. When in later life responsibilities crowded upon him, and his friends were amazed at the amount of work of different kinds that he could get through, nothing was allowed to diminish the care and attention he gave to his teaching work.

It is appropriate that in this place special emphasis should be given to his work as a teacher of mathematics, but any adequate account of his life and work would have to include other aspects of the activity of this many-sided man. It would have to speak of his distinguished service to the University of London. That the old antagonism between its Internal and External sides is now but an unhappy memory of a receding past is due very largely to his statesmanship and his ability to work hard, long, and patiently to achieve a purpose he had set before himself. It would have to speak too of his important contributions to the advance of his subject, notably by his researches in the Theory of Elasticity. Throughout his academic life, even when other claims on his time and energy were most insistent, he never forgot his duty as a professor to extend the knowledge of his subject by his own researches.

But to those who knew him best and worked most closely to him, it is the memory of the man himself that will be most reverently cherished. He had a spirit that was matched to the dimensions of his physical frame. He was a man of steady loyalties and simple honesty. He had strong convictions and held them strongly. He could so easily have been intolerant if it had not been that his deepest conviction of all was of the value of freedom and of liberty. Behind it all, known in a measure to many but known to the full by the few, there was a great spirit of simple kindliness and unfailing loyalty. If Filon was your friend, you knew that his was a friendship that would stand in fair weather or in foul, and that at its call he would give of himself generously and to the full. G. B. J.

ALFRED LODGE.

ALFRED LODGE became a member of the A.I.G.T. in 1888, was Honorary Treasurer from 1891 to 1896, and first President of the Mathematical Association in 1897-1898. Until recently he was constant in his attendance at meetings of the Council; we shall not easily forget his keen interest in all the activities of the Association or the charm of his kindly manner. His first contribution to the *Gazette* appeared in No. 1; his last in No. 239.

After a distinguished career at Oxford, including a Fereday Fellowship at St. John's College, Lodge was from 1884 to 1904 Professor of Pure Mathematics at the Royal Indian Engineering College, Coopers Hill, and then for fifteen years mathematical master at Charterhouse. A former Charterhouse colleague, Mr. C. O. Tuckey, contributes the following inspiring account of mathematical thought continued in the closing years of a long life.

I wonder whether there are friends of Alfred Lodge who would be interested to read of his mathematical activities a month or so before his death, and whether the younger readers will care to know that mathematics can be a keen interest and a solace to a bereaved and lonely old man in his 84th year.

Lodge was my colleague for fifteen years in mathematical specialist teaching at Charterhouse, and after his retirement we corresponded regularly on all sorts of mathematical questions. When I say regularly, he was much more regular than I was; he always answered my letters promptly, which is more than could be said for me. I believe he also corresponded similarly with various other friends.

But to his recent activities; this summer he was interested in theory of numbers. For the problem "To show that N^{10} is either $41n$, $41n \pm 1$, or $41n \pm 9$" he sent me this solution.

"By Fermat, if N is prime to 41, N^{40} is of form $41n+1$;

$\therefore$ $(N^{20}-1)(N^{20}+1)$ is $M(41)$, i.e. $N^{20} = 41n \pm 1$;

$\therefore$ N^{10} is such that its square is $41n \pm 1$.

If $N^{10} = 41n \pm 1$, its square is $41n+1$, and in no other case.

If $N^{10} = 41n \pm 9$, its square is $41n+81$, i.e. of form $41n-1$.

All I had to do was to take all the squares from 1 to 20 inclusive, divide them by 41, and see which gave the required remainder -1. I found that 9^2 did it. When these squares are divided by 41 all the remainders are different up to 20^2; after that they repeat backwards, since r^2 and $(41-r)^2$ leave same remainder.

So the solution is unique; 9^2 is the only chance.

Hence N^{10} is $41n$, or $41n \pm 1$ or $41n \pm 9$.

This makes N^{20} of form $41n$, $41n+1$, $41n-1$,

and N^{40} of form $41n$ or $41n+1$."

This October, *à propos* of something of mine on Mass and Weight in the *Gazette* :

" For beginners I am convinced that Newton's second law is best given as $P/W = f/g$, i.e. accelerations are proportional to the forces acting on a given body, actually or potentially. The equality $P/W = f/g =$ constant for a given body is more abstruse in its ideas and should be delayed a bit. $W/g = M$ is chiefly a matter of *quality*. W and g are both vertical vectors ; M is a scalar quite directionless. Important but too advanced for youngsters : that is one reason for delaying a dissertation on M."

This November we discussed the problem : " By how many routes can a King on an empty chessboard go from his usual starting square to a square on the opposite side of the board? " We neither of us reached a formula capable of extension to a board of n^2 squares, but this is what he sent me about a month before his death.

" On an unrestricted board the numbers of routes to the various squares are the coefficients in $(x^{-1}+1+x)^n$

The actual board showing the losses in red [printed in italics] :

			1					
		1	1	1				
	1	2	3	2	1			
	1	3	6	7	6	3	1	
1	4	10	16	19	16	10	4	1
1 5	14 1	30	45	51	45	30	15	5 *1*
1 6 21	44 6	89 1	126	141	126	90	50	20 1 *6 1*
1 7 28 77	133 28	259 7	356 1	393	357	266	160 1	70 7 *28 7 1*

We lose

$$x^7 + 7x^6 + 28x^5 + 77x^4 + 28x^3 + 7x^2 + x + x^{-3} + 7x^{-4} + 28x^{-5} + 7x^{-6} + x^{-7}.$$

The total is $3^7 - 149 - 44 = 1994$."

May all members of the Association, whether the problems with which they are concerned are more abstruse than these or less so, find like Alfred Lodge that their interest in them never falters.

C. O. TUCKEY.

THE MATHEMATICAL ASSOCIATION.

THE Annual Meeting of the Mathematical Association was held at the Institute of Education on 4th and 5th January, 1938. On Tuesday, 4th January, Professor E. H. Neville took the chair for the Business Meeting at 2.30 p.m. Professor Neville spoke of the loss suffered by the Association through the death of the President, Professor L. N. G. Filon, on 29th December.

The Report of the Council for 1937 was adopted.*

The Hon. Treasurer presented a statement of accounts for the year ending 31st October, 1937.

On the nomination of the Council, Mr. W. Hope-Jones was elected President for the year 1938. The existing Vice-Presidents, Treasurer, Secretaries, Librarian and Editor of the *Gazette* were re-elected, except that Professor A. R. Forsyth retired from the office of Vice-President at his own request. Miss G. K. Stanley and Mr. C. T. Daltry retired from the Council and were not eligible for re-election. The following were elected as members of Council for 1938:

Miss M. J. Griffith,
Professor G. B. Jeffery,
Mr. W. J. Langford,
Miss W. M. Lehfeldt,
Mr. C. W. Parkes,
Mr. A. W. Riley,
Mrs. E. M. Williams,
Miss M. A. Hooke,
Mr. F. J. Swan.

The Presidential Address, *Mass and Weight in Newtonian Mechanics*,† which had been prepared by Professor Filon, was read by Professor G. B. Jeffery. This was followed by Professor D. R. Hartree's lecture: *The Mechanical Integration of Differential Equations.*

On Wednesday, 5th January, with Professor E. H. Neville in the chair, the proceedings opened with a discussion: *The Relative Value of Pure and Applied Mathematics*‡ introduced by Dr. W. G. Bickley, Mr. C. G. Nobbs and Miss K. I. Sayers. This was followed by Mr. N. R. C. Dockeray's paper, *The Law of Quadratic Reciprocity*,‡ and Mr. N. M. Gibbins' paper, *The Feuerbach Quadrilateral.*‡

The afternoon meetings opened with Mr. M. Black's paper, *The Relevance of Mathematical Philosophy to Mathematical Teaching*,‡ and concluded with a discussion: *Teaching the Complete Duffer*,‡ introduced by Mr. B. L. Gimson, Mr. F. C. Boon and Dr. F. H. Dodd.

A Publishers' Exhibition was open during the two days.

* See pp. 6-8. † See pp. 9-16. ‡ To be published later.

GLEANINGS FAR AND NEAR.

1171. "Il n'y a que ceux qui ne calculent pas, qui ne se trompent jamais." Le Verrier.—Giovanni Boccardi, *Guide du Calculateur*, p. 70 (Paris et Catane, 1902). [Per Mr. Frank Robbins.]

REPORT OF THE COUNCIL FOR THE YEAR 1937.

DURING the year 1937, 153 new members have joined the Association, of whom 40 are junior members. The number of members now on the roll is 1648, of whom 9 are honorary members, 108 are life members, 1463 are ordinary members and 68 are junior members.

The Council reports with the deepest regret the untimely death of the President, Professor L. N. G. Filon, and wishes to record its appreciation of the services rendered by him to the Association.

The Council regrets to have to report the deaths of the following members of the Association: Professor E. B. Elliott, Mr. G. H. Fothergill, Dr. S. F. Grace, Rev. T. T. Griffiths, Professor Alfred Lodge, Professor E. B. Lytle, Dr. F. S. Macaulay, Mr. J. M. Reay and Mr. D. C. Tayler.

Professor Lodge had been a member of the Association for almost 50 years. He was Treasurer from 1891 to 1896 and President in 1897 and 1898. As a Vice-President since 1899 he had regularly attended Council Meetings and had throughout his long connexion with the Association shown a keen interest in all its activities, especially those of the early Teaching Committees.

Dr. Macaulay had been a member of the Association for 50 years, and during the earlier part of that time he played an important part in its activities. He was a member of the Council for many years and for two years he edited the *Mathematical Gazette*.

The Branches.

The Branches all show in their reports a continuance of vigorous activity and membership has been well maintained. Lists of their officers and committees are included in the List of Members recently published, and particulars of their meetings have been given in insets in the *Gazette*. The Manchester Branch suffered a severe loss last winter in the death of its President, Mr. Doughty, who had been a source of great help and inspiration to its members. The Liverpool Branch has, with the co-operation of the local Education Authority, circularised schools of various types with a view both to widening its scope and increasing its numbers; the results so far have been very satisfactory. The Southampton Branch has recorded its unanimous opinion as to the depressing effect of the London Matriculation Syllabus on geometrical teaching.

The composition of the Branches in terms of members, associates and junior members is as follows: Bristol, 15 members, 21 associates; Cardiff, 39 members, 48 associates, 1 junior member; Liverpool, 16, 38, 6; London, 190, 90; Manchester and District, 36, 70; Midland, 37, 35; North-Eastern, 35, 15, 2; Northern Ireland, 4, 32; North Wales, 5, 14; Southampton and District, 6, 20, 3; South-West Wales, 11, 29; Yorkshire, 42, 78, 1; Queensland, 11, 16; Sydney, New South Wales, 21, 129; Victoria, 9, 16.

The Mathematical Gazette.

The May number contained an article, written by Professor J. Devisme at the request of the Editor, comparing the teaching of analytical trigonometry in France with its teaching in England. It is hoped that further articles may be secured, comparing the teaching of specific subjects in England with that in other countries.

An extra number of the *Gazette*, published in November, contained reports of discussions on the teaching of Algebra and Geometry and also a paper on "Hypersolid Concepts". These took place at the meetings of Section A* of the British Association, held at Nottingham in September, 1937.

In October, 1937, Professor C. J. de la Vallée Poussin, of the University of Louvain, gave two lectures at University College, London. These lectures will be published in Vol. XXII of the *Gazette*.

The Library.

The Library continues its postal service to members and there has been no falling off in the number of those who make use of it. On the other hand there has not been the increase in the number that might have been expected, in view of the steady growth of the membership of the Association, and a leaflet calling the attention of members to the Library was issued with the December *Gazette*.

The General Teaching Committee.

The General Teaching Committee has approved for publication "A Second Report on the Teaching of Geometry in Schools" which has been prepared by the Boys' Schools Committee. It is hoped that the new report will be in the hands of members of the Association by February. The sales of the Arithmetic report have been such as to call for a reprint during the last few months.

The Boys' Schools Committee.

The Boys' Schools Committee has continued during the year its work on the report on the teaching of Geometry mentioned above. The full text was approved at a meeting held on October 2nd, and the report was then referred to the General Teaching Committee.

This report is intended to supplement and not to replace the report of 1923, the sales of which are well maintained.

The Girls' Schools Committee.

The Girls' Schools Committee held one meeting this year. Some aspects of the teaching of Algebra were discussed.

The Problem Bureau.

The Problem Bureau continues to render valuable service to members, especially to those whose administrative duties make it difficult for them to devote sufficient time to the "difficulties" which arise in teaching scholarship candidates. About 50 solutions

per annum are sent out, the majority being of questions in Calculus and Mechanics.

The Secretary of the Bureau wishes to renew his thanks to those members who have assisted him, especially Mr. G. A. Garreau for his valuable help in the routine work and Mr. W. J. Hodgetts for the elegance of his solutions to questions in Pure Geometry.

General.

The arrangements made by the Council to rent a room in Gordon House and to make this a centre for some of the routine work of the Association have now been completed and are working in a very satisfactory manner.

The new Teaching Committee (appointed under the revised regulations passed at the last Annual Meeting) has been appointed by the Council. A list of the members appointed has been circulated.

The Council takes this opportunity of thanking all the members of the retiring Teaching Committees, especially the Chairman and Secretary of the Boys' Schools Committee who have rendered valuable service in connection with the new Geometry Report.

The Council recently appointed a sub-committee to enquire into the question of the affiliation of student mathematical societies to the Association. It was felt in general that if such affiliation proved practicable it might well prove to be of great assistance in attracting new members.

The new List of Members has been issued. In accordance with a decision of the Council, an Index of Members according to the schools in which they work, has been substituted for the Topographical Index which has appeared in past lists.

Officers and the Council.

The Council nominates Mr. W. Hope-Jones as President for 1938.

The Council also desires to record its appreciation of the very valuable work done by Mr. T. A. A. Broadbent as editor of the *Mathematical Gazette*, by Professor E. H. Neville as Librarian of the Association, and by Mr. A. S. Gosset Tanner in connection with the Problem Bureau.

The thanks of the Council are also due to Miss G. K. Stanley and Mr. C. T. Daltry, retiring members of the Council.

1172. Sir John Simon, Chancellor of the Exchequer, told students of the Working Men's College . . . " One of the busiest King's Counsel of the last generation, in the interstices of an absorbing practice, used to work through the whole of the Differential Calculus* with all the exercises of a standard text-book, every year ". The following footnote is added :

* Differential calculus is incredibly intricate mathematical method to solve problems involving moving bodies such as the speed of a hurled ball gradually coming to rest under the influence of gravity.—*Sunday Referee*, 24th October, 1937. [Per Dr. N. W. McLachlan.]

MASS AND FORCE IN NEWTONIAN MECHANICS.

By Professor L. N. G. Filon.

(*Presidential Address to the Mathematical Association, January* 1938.*)

The considerations which I propose to put before you to-day arose from an attempt, when teaching students, to build up the classical laws of Newtonian Mechanics inductively from laboratory experiments.

In this task we are immediately confronted with two fundamental notions, those of *mass* and *force*.

It will probably be admitted that no definition of a physical quantity can be accepted as adequate unless it implies a criterion leading to an unambiguous measure of the thing defined.

Leaving out of account empirical measures of force such as that given by the spring balance, let us see what criterion is provided by the three laws of motion which lie at the base of Mechanics.

Let us begin with the First Law, which states, in effect, that if a body P is under no force, then it has no acceleration.

We must, however, define an origin O with respect to which the acceleration of P is measured. Newton took this origin in some fixed absolute space. But as it is impossible to locate physically such a point, we can turn the difficulty by saying that, if we have two bodies O and P each under no force, P has no acceleration with regard to O. If we take O as our origin the First Law ought to hold for motion relative to O, and we may call O a Newtonian origin.

The question then arises: When is a body under no force? So far as our experience goes the only forces which directly affect our senses are those acting by contact. We are not sensually aware of what we term weight, but only of the external or internal reactions which prevent our body or its parts from falling freely. If we restrict forces to what may be called *contact* forces, then a free body should be under no force.

If, however, we take as O and P two free bodies at antipodal points of the Earth, say here and in New Zealand, we find that they both accelerate towards the centre of the Earth, and therefore they have an acceleration relative to one another.

Thus we must either reject Newton's First Law, or introduce some hypothetical intangible forces which will account for the accelerations observed. Such a force I will term a *field* force, and the simplest example of such is the familiar force of weight.

We must, however, remember that any restriction on these hypothetical forces as regards their dependence on time, position, velocity and acceleration, amounts to the introduction of an independent law of Mechanics, and should be avoided until it is shown to be absolutely necessary.

Two important comments may be made on this.

*The President died on 29th December, 1937; this address was read at the Annual Meeting on 4th January, 1938 by Professor G. B. Jeffery.

First of all, we can now never know when a body is under no force, and therefore our criterion for a Newtonian origin disappears.

Secondly, by introducing a suitable field force, any body O, moving in any manner, may be regarded as a Newtonian origin.

The First Law can therefore never be proved or disproved by any experiment.

Now take the Second Law. In its modernised form it runs : the force on a body is proportional to its mass multiplied by its acceleration. No loss of generality, as is well known, is involved if we replace in this the word " proportional " by the word " equal ", this being merely a question of choice of units. The acceleration here is the acceleration relative to a Newtonian origin, but we may eliminate this difficulty if we replace the Second Law by its well-known corollary that every force produces its own acceleration. In other words, an additional force acting on a body is equal to the mass multiplied by the additional acceleration produced. This additional acceleration is independent of the choice of origin to which the motion is referred.

Here we come up against mass for the first time. If we enquire what mass is, and how it is to be measured, we find mass described by Newton in the first paragraph of the *Principia* as the quantity of matter in a body. Assuming the principle of the conservation of matter, we may deduce from this that mass is something which is invariable for a given body. But it is still not a definition, for we have been given no means of measuring it. Newton clearly sees this, and goes on to explain that it is the product of volume multiplied by density. At this point one suspects that he saw difficulties, for he says " the ratio of all bodies remains the same when they are condensed differently from any causes whatsoever ". He then pursues the question of density no further, but makes the following statement : " This quantity of matter will be denoted in what follows by body or mass. It is obtained from the weight of each body. For it is found to be proportional to the weight, according to the most accurate pendulum experiments, as will be shown later."

In other words, Newton, after talking round the question, defines mass by weight. Thereby he does give a means of measuring it, for eventually the theory of mechanics leads to a method of comparing weights. But, in the first place, his definition anticipates his conclusions, and, in the second place, it makes the laws of motion dependent upon gravity, that is upon the field force at the earth's surface, yet it has always, since Newton, been argued that the law of gravitation was something additional to, and independent of, the laws of motion.

Curiously enough, this reference to weight in the *Principia* seems to have been almost universally ignored, and for over two centuries we find the purely verbal definition of mass by " quantity of matter " repeated without his explanation. If now we proceed in logical

order, and leave weight out of consideration for the present, "mass" in the Second Law is merely an undetermined constant belonging to a body, and it appears that all the Second Law can tell us is how to compare forces which are acting on the *same* body.

In 1872 Ernst Mach published his *Science of Mechanics* in which, for the first time, the foundations of the subject were critically examined.

Mach pointed out that the Third Law (action and reaction are equal and opposite) provided a means of comparing forces acting on different bodies, and that the accelerations produced by the mutual reactions of two bodies must be inversely as their masses. This he held to give a direct means of comparing masses, so that the Second and Third Laws together provide a complete definition of force and mass.

Although I shall refer to such an experiment as Mach's experiment, Mach himself never attempted to carry it out. It is not, however, difficult to do so, and there are several variants of it, but, in order to avoid long-winded complications, I shall here mention a form which is comparatively simple in theory, though unsuitable in practice.

Two bodies A and B, connected by a light stretched spring or elastic string, are held apart at the same level and simultaneously released, and their paths traced. After a time they would, if unconnected, have fallen to A', B'. If their actual positions are P, Q, then $A'P$, $B'Q$ are their simultaneous displacements due to the

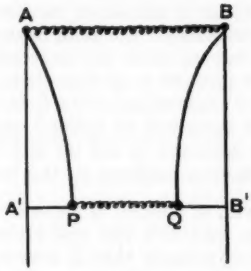

FIG. 1.

mutual action transmitted through the spring. These displacements are found to be in a constant ratio, and therefore the additional accelerations are in the same ratio, which gives the inverse ratio of the masses.

This argument of Mach's has never percolated to any extent into the text-books, but it has been accepted very generally by the experts. Karl Pearson gives it his entire approval in his *Grammar of Science*, and, until some five years ago, I was myself under the impression that it was valid.

I believe now that Mach's argument is fallacious for the following reason.

Just as we can never know when a single body is under no force, so we can never know when two bodies are moving solely under their mutual reactions.

Thus, in the experiment described, if we can suppose that the only field force acting on each body remains vertical, Mach's argument is valid. But if, owing to the deflection of the bodies, a new field force having a horizontal component is introduced, no conclusion can be drawn.

The experiment in a modified form does actually give a definite ratio, but it will appear later that this ratio has not the significance that Mach attached to it.

Further, the Second and Third Laws, like the First, can never be contradicted by experiment, for whatever the result of such an experiment, we can always invoke a suitable "field force" to explain the discrepancy.

It appears then that at this stage the three laws of motion are not experimental laws. They cannot be proved by direct experiment. On the other hand, as they cannot be disproved either, it is convenient to assume them, *leaving mass undetermined*, and to proceed, until we are in a position to explore experimentally the field of force at the Earth's surface.

We can now develop the laws of Mechanics in the usual manner, obtaining D'Alembert's Principle, the equations of motion of solid bodies, the properties of light strings and pulleys, and the principle of the lever. The object of this is to enable us to use the simple mechanisms of the laboratory. We shall meet on the way a number of other difficulties, but as these are independent of the question now at issue, I do not propose to go into them.

We now come to the exploration of the field of force at the Earth's surface. This field of force will be defined quite simply as the field force which must be assumed to act on any body at any time, to make the observed motion conform to the laws of motion. This I shall call $\vec{W}$, and it will, of course, in general, be a vector quantity.

Our first step is to establish the well-known law governing the fall of a body *in vacuo*, namely that it moves, at a given place on the Earth's surface, with a constant acceleration g directed along the downwards vertical, defined by a plumbline at rest. We have then, for such a freely falling body,

$$W = mg. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

We must, however, avoid carefully the use of any diluting apparatus, such as the inclined plane, Atwood's machine or the pendulum, which involves constraint. There are, however, modern laboratory methods which enable us to measure g directly, and we will suppose this has been done.

But we must not assume now, as is always done, that the W

keeps the same value, either in magnitude or in direction, when the body is constrained to move in a prescribed manner.

For example, if the same body is suspended by a light string from a fixed support, we know that it hangs vertically. In this case the tension T ($=T_1$, say) balances W, which must then act vertically downward. The corresponding value of W we shall term the *rest-weight* of the body and denote it by w. Note carefully that w is not necessarily equal to mg.

By means of the lever or balance we can compare rest-weights, though we cannot determine their absolute values, and we can measure forces in terms of rest-weights.

Suppose now we allow our point of support to move vertically with a vertical acceleration f, measured positive downwards, observation shows that the string remains vertical. Thus W, which balances the tension T and the reversed mass-acceleration, is vertically downward and equal to

$$T+mf.$$

To measure T we use a modification of an experiment due to Poggendorff, in which an Atwood's machine consisting of a light pulley P, over which is hung a light string connecting two bodies A and B, is suspended from one arm of a balance and weighed against a scale pan Q containing weights suspended from the other arm, both when the pulley is clamped, so that A and B are stationary, and when they are free to move. The difference of the weights in the scale pan in the two cases is $w+w'-2T$, w and w' being the rest weights of A and B found by weighing with the balance. Hence T is determined. Of the two bodies, we fix attention on one,

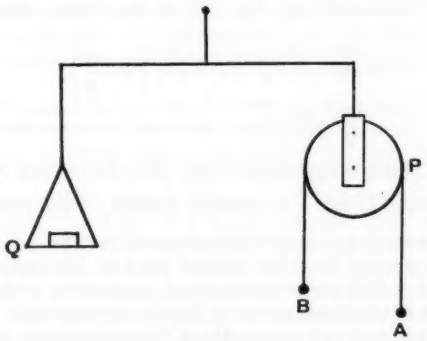

FIG. 2.

A, of mass m and rest-weight w. The other, B, is merely used to control the acceleration f, and actuates a recording apparatus which enables f to be measured.

If the experiment is carried out carefully, with different bodies B, but the same body A throughout, and T is plotted against f, the resulting straight line graph found shows that $T_1 - T$ is proportional to f, being equal to λf, where λ is, within the errors of observation, w/g, g being the quantity previously determined ; λ is independent of position, velocity or time.

Hence

$$W = T + mf = T_1 + (m - \lambda)f = w + (m - \lambda)f, \quad \ldots\ldots\ldots\ldots(2)$$

since T_1, as before, must be the rest-weight of A.

That $\lambda = \dfrac{w}{g}$ follows by writing $f = g$ in (2) and using (1). Accordingly for vertical motion with downward acceleration f,

$$W = w + \left(m - \frac{w}{g}\right)f. \quad \ldots\ldots\ldots\ldots(3)$$

Experiments with an inclined plane and pulley, in which one body hanging vertically is connected with a heavy trolley on light frictionless wheels moving up or down the inclined plane, enable us to pass from W under vertical acceleration to W under oblique (or as a special case) horizontal acceleration. The results lead to the conclusion that, in all cases, the field force which must be assumed is obtained by compounding with the rest-weight a force

$$\left(m - \frac{w}{g}\right)f, \quad \ldots\ldots\ldots\ldots(4)$$

proportional to the acceleration and acting in the same direction.

Once this result has been arrived at, it is clear that no experiment can possibly contradict it. For if $\vec{F}$ is the contact-force acting on the body,

$$\vec{mf} = \vec{F} + \vec{W} = \vec{F} + \vec{w} + \left(m - \frac{w}{g}\right)\vec{f}$$

or

$$\frac{w}{g}\vec{f} = \vec{F} + \vec{w}, \quad \ldots\ldots\ldots\ldots(5)$$

that is, everything happens as if the field force were always equal to the rest-weight, but $\dfrac{w}{g}$ is written instead of the mass. This we know always leads to correct experimental results.

It will be noticed that the inertial mass m has now disappeared from (5), and is still undetermined and apparently undeterminable.

What now is the significance of Mach's experiment?

Clearly it can only give the ratio of the *rest-weights*, which we can get more simply from the ordinary balance.

Accordingly Mach's procedure for determining the ratio of the inertial masses m without introducing gravity is illusory, and we have, in fact, been brought back, by a devious route, to Newton's definition, which appears to be the only possible one.

It may be asked, why not assume $w/g = m$, and get rid of these awkward additional terms? The reason is that this law of field force has a very simple physical interpretation. A spherical body moving in an infinite incompressible fluid with acceleration $\vec{f}$ is known to experience a resistance $m'\vec{f}$ where m' depends on the volume of the body and the density of the fluid. Thus if $\vec{F}$ be the actual force acting on it,

$$m\vec{f} = \vec{F} - m'\vec{f}, \quad \text{i.e. } (m+m')\vec{f} = \vec{F},$$

or the body moves as if its mass were $m + m'$.

But this is precisely the law we have found with $m' = \frac{w}{g} - m$. If all bodies at the Earth's surface were bathed in such a fluid, not unlike the nineteenth century ether, we could never become aware of it by any dynamical experiment, and the only mass we could know would be the effective mass $m + m'$ or w/g.

But this effective mass will vary if the density of the fluid is different in different places; it no longer belongs, like the hypothetical inertial mass, to the body and to the body alone. May not something of the same kind happen to w/g?

We know, from pendulum experiments, that g varies both with height and depth and with position on the Earth. Does w vary in the same ratio?

This question is not an easy one to answer. We know from accurate balance experiments that the ratio of the rest-weights of two bodies is invariable, so that if the question could be answered for one body, it would be answered for all.

A direct comparison of rest-weights at two different places, although theoretically possible, presents insuperable practical difficulties.

An indirect comparison could be made by weighing a body in an accurate spring balance, first at, say, London and then at the Equator. But a much better experiment of this class is daily carried out unconsciously by the navigator of every ship which carries a chronometer. If w/g varied at all, the rate of the chronometer would vary, and this is a highly delicate test. It is curious that this observation which, so far as I am aware, provides the only experimental evidence for the validity of the concept of an absolute effective mass, is never even mentioned in text-books on Mechanics.

It will be seen that the argument here involves the assumption that the "spring balance" measure of force is invariable and absolute. This amounts to a new axiom of Mechanics, which so far we have not found necessary.

There is, however, another way of turning the difficulty, which is simply to assume the result. For, if we disregard the spring balance, we have seen that the absolute value of w at any given place is not

determined, and we may arbitrarily assign to it the value g for some standard body, which will be our unit of mass. It is true that we have no *theoretical* certainty that this will not some day be contradicted by experiment, but we have at any rate a *practical* certainty at the present time and the assumption is in agreement with the chronometer result, for what it is worth.

I cannot to-day enter into the question of celestial observations, except to say that careful analysis of their implications does not apparently disturb our conclusions.

I hope that this discussion of fundamental ideas will not appear to members of the Association too abstract or too remote from the problems of practical teaching. I think, however, that it may be of value for teachers to have some notion of what I may call the inner doctrine of their subject-matter, even though they wisely refrain from introducing it to beginners.

It seems at any rate remarkable that after all the fuss we have made about the fundamental distinction between mass and weight and the criticisms levelled at the engineer for his systematic use of w/g, it should turn out that the engineer was right after all, and, incidentally, that Newton's amazing intuition had, in that disregarded definition, hit upon the inevitable truth.

1173. Every step forward puts the end further away.—R. A. Sampson, *The Sun*, p. 7 (Cambridge, 1914). [Per Mr. Frank Robbins.]

1174. "*Nein! Wir sind Dichter*"—Kronecker in Berlin. Such were the pregnant words recently uttered by the youngest of the splendid triumvirate of Berlin, when challenged to declare if he still held the opinion advanced in his early inaugural thesis (to the effect that mathematics consists exclusively in the setting out of self-evident truths—in fact, amounts to no more than showing that two and two make four), and maintained unflinchingly by him in the face of the elegant raillery of the late M. Duhamel at a dinner in Paris, where his interrogator—the writer of these lines—was present.—Sylvester, *Mathematical Papers*, vol. iv (1912), p. 625. [Per Prof. H. G. Forder.]

1175. Homer Nods.

I thought I had seen something about the number of umbilici . . . Voss . . . (1875) gives the number for a surface of the nth order as $=n(10n^2-28n+22)$, instead of Salmon's $n(10n^2-25n+16)$, viz. Salmon's number is reduced by $12(n-2)$, and in the case $n=4$ it is $=280$ instead of 304.—Cayley to Stokes, *Memoir and Correspondence of Stokes*, I, p. 396 (1907). Salmon himself, or Cathcart on his behalf, makes the correction correctly: compare the footnotes on p. 237 of 3rd ed. (1874) and p. 263 of 4th ed. (1882) of his *Geometry of Three Dimensions*. Cayley's letter is dated only 14th Nov. and is placed between letters dated 8th March [1873?] and 26th Sep. 1874, implying the year 1873, in spite of the date attached in the letter itself to Voss's name, but it is hard to believe that the edition of Salmon's book published in 1874 contained a mistake of which Cayley, who is said in the preface to have read all the proof-sheets, was aware in the previous year. [Per E. H. N.]

POTENTIEL ET PROBLÈME GÉNÉRALISÉ DE DIRICHLET.*

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CONFÉRENCES FAITES À L'UNIVERSITÉ DE LONDRES LES 12 ET 14 OCTOBRE 1937.†

1. CONSIDÉRATIONS PRÉLIMINAIRES. BALAYAGE ; PROBLÈME SIMPLE ET PROBLÈME GÉNÉRALISÉ DE DIRICHLET.

On connaît le problème classique ou le problème simple de Dirichlet. On donne une surface fermée S qui partage l'espace en deux domaines, l'un intérieur, l'autre extérieur. On suppose que la surface satisfait à certaines conditions de régularité, par exemple, être douée d'un plan tangent. On assigne une fonction continue $U(M)$ d'un point M sur la surface S, et l'on demande de déterminer la fonction $U(P)$, harmonique et bornée dans l'un des deux domaines précédents, qui tend vers $U(M)$ quand le point P tend vers le point M de la frontière. Le problème est intérieur ou extérieur selon le domaine considéré.

Dans un Mémoire étendu que j'ai publié dans les *Annales de l'Institut H. Poincaré* de 1932,‡ j'ai montré que ce problème se résout par une opération que j'ai appelée un balayage, et ce nom lui est resté depuis.

S'il s'agit du problème intérieur, ce sera un *balayage intérieur* et voici en quoi cela consiste. On place l'unité de masse au point P du domaine considéré ; le balayage a pour effet de remplacer cette masse unité par une couche répartie sur la surface S sans changer le potentiel ni sur S ni à l'extérieur.§ Dans le Mémoire cité, je n'ai fait qu'indiquer un procédé pour déterminer cette couche qui s'inspire de la méthode dite du balayage de Poincaré, méthode que le *Traité d'Analyse* de M. E. Picard a rendue classique, mais cette opération n'est faite en aucune façon par H. Poincaré.

Quand la distribution μ de la couche balayée sur S est connue, le problème de Dirichlet est immédiatement résolu par l'intégrale suivante, à laquelle nous donnons la forme de Stieltjes

$$U(P) = \int_S U(M) d\mu(M).$$

* Nous avons traité le même sujet dans des conférences que nous avons eu l'honneur de faire à Strasbourg en décembre 1936, mais nous le reprenons ici avec des démonstrations nouvelles et plus directes, que nous rattachons toutes à la considération de la même intégrale définie. Par contre, nous avons exposé à Strasbourg un important élargissement du problème de Dirichlet qui sera laissé de côté dans cet article-ci. Nos conférences de Strasbourg sont publiées dans les *Publications de l'Institut Mathématique de Strasbourg*, II. Paris, Hermann, 1937.

† [The thanks of the Editor are due to Professor de la Vallée Poussin and to the University of London for permission to publish these lectures in the *Gazette*.]

‡ " Extension de la méthode du balayage de Poincaré et problème de Dirichlet ", Paris, 1932.

§ Cette opération n'était nullement nouvelle, elle avait déjà été faite par Gauss, il y a près d'un siècle.

On prouve que $U(P)$ est une fonction harmonique de P par une démonstration que nous reprendrons plus loin. On prouve que $U(P)$ tend vers $U(M)$ quand P tends vers M en montrant que toute la masse μ balayée sur S, égale à l'unité, vient à la limite se concentrer au point M. La démonstration en est presque immédiate si la surface S admet un plan tangent.

Mais si les conditions de régularité classiques viennent à manquer, comme c'est le cas pour un domaine ouvert dont la frontière est quelconque, le problème de Dirichlet n'admet plus en général de solution rigoureuse et c'est M. Lebesgue qui en a donné le premier exemple. Une fonction harmonique dans le domaine ne tendra pas vers une fonction continue à la frontière. Il est donc naturel de se demander s'il est possible d'élargir le problème de Dirichlet de manière à lui assurer encore une solution et une seule. C'est la question du *problème généralisé de Dirichlet*, que M. Norbert Wiener a posée en 1924, mais qui a résisté pendant près de dix ans à ses efforts, faute de moyens appropriés pour le résoudre.

Il n'y avait aucune difficulté à construire une fonction, harmonique dans le domaine ouvert, satisfaisant aux conditions à la frontière à l'exception de certains points de celle-ci, dits *irréguliers*, et ne dépendant que de cette frontière seule, mais c'était vainement que l'on tentait d'assigner à ces points irréguliers un caractère suffisant d'exception pour en pouvoir déduire l'unicité de la solution.

Il fallait, pour y arriver, employer les méthodes nouvelles de la théorie du potentiel dont j'ai fait un usage systématique dans le Mémoire cité, et qui consiste à définir la répartition des masses, non plus par la densité qui est une fonction de point, mais par une fonction d'ensemble borélien précisant la masse portée par un ensemble quelconque de points. Il fallait aussi une notion précise capable de caractériser le degré de rareté des points irréguliers, c'est celle d'ensemble de capacité nulle. Elle permet ce simple énoncé : les points irréguliers forment un ensemble de capacité nulle. Cet énoncé est équivalent à un énoncé antérieur, mais indirect, connu sous le nom de lemme de Kellogg.

Le problème généralisé a été finalement résolu, grâce aux efforts de divers mathématiciens distingués, parmi lesquels il convient de signaler particulièrement Kellogg, Evans, Bouligand et Vasilesco. Mais nous ne ferons pas ici l'histoire de leurs recherches, nous renverrons pour cela à une étude récente et excellente de M. Vasilesco, couronné par l'Académie Royale de Belgique.* Nous nous proposons de traiter ici le problème généralisé de Dirichlet par une voie beaucoup plus directe que celle suivie par les auteurs précédents. Nous en rattacherons la solution à la condition de minimer une certaine intégrale définie considérée par Gauss en 1840. Le mérite d'être revenu à cette intégrale revient à M. Frost-

* Le problème généralisé de Dirichlet, par Florin Vasilesco. *Mémoires in* 8° *de l'Académie Royale de Belgique*, t. xvi, fasc. 4, 1937.

man.* La méthode de Gauss, reprise d'ailleurs par Dirichlet, a été discréditée par Weierstrass, qui lui a reproché de supposer l'existence d'un minimum sans la démontrer, et l'on sait par combien de détours on a tenté au 19e siècle d'éviter cette démonstration. M. Frostman en revenant aux méthodes de Gauss les a généralisées et rendues rigoureuses.

Nous allons rattacher, dans les pages suivantes, la théorie du balayage et les propriétés de l'intégrale d'énergie aux conditions de minimum de l'intégrale de Gauss et cela d'une façon plus générale et plus exclusive que nous ne l'avons fait dans nos conférences antérieures à Strasbourg. Cela nous donnera aussi quelques résultats nouveaux. Nous commencerons par rappeler brièvement la définition généralisée du potentiel et quelques propriétés fondamentales qui résultent de son expression par une intégrale généralisée de Stieltjes. On peut en trouver les démonstrations dans mon Mémoire de 1932, dans l'ouvrage cité de M. Frostman, ou dans la *Théorie de l'Intégrale* de Stanislaw Saks.† Nous montrerons enfin que le problème généralisé de Dirichlet se résout par la même intégrale de Stieltjes que le problème simple et que la justification en est donnée par le même moyen, esquissé plus haut.

2. Définition du potentiel.

La distribution des masses se définit, sans aucune hypothèse relative à la densité, à l'aide d'une fonction bornée et additive d'ensemble borélien μ.‡ La valeur de μ sur un ensemble e est la charge positive ou négative portée par e. Nous disons que la distribution définie par cette fonction est la *distribution* μ et que les masses qu'elle distribue sont les *masses* μ.

Le *noyau de masses* pour une distribution μ est l'ensemble des points non distants des masses. Un point du noyau est donc le centre d'une sphère contenant des masses quelque petit qu'en soit le rayon. Le noyau est un ensemble fermé.

Soit μ une distribution de masses sur un ensemble E. Son potentiel U sur un point P extérieur au noyau se définit par l'*intégrale généralisée de Stieltjes*

$$U(P)=\int_E \frac{d\mu(Q)}{PQ},$$

et l'on dit que c'est une intégration *par rapport aux masses*. Une telle intégrale se définit en remplaçant, dans la définition d'une intégrale de Lebesgue, la mesure des ensembles par les charges qu'ils portent. Il est immédiat que *le potentiel, qui est une fonction du point P, est continu hors du noyau.*

* Potentiel d'équilibre et capacité des ensembles avec quelques applications à la théorie des fonctions, par Otto Frostman. *Meddelanden fran Lunds Universitets matematiska seminarium*, Bd. 3. Lund, 1935. C. W. K. Gleberup.

† Monografje matematyczne, t. II. Warsawa, 1933. 2e éd. anglaise, 1937.

‡ Pour la théorie des fonctions d'ensemble, nous renvoyons à notre ouvrage de la Collection Borel : *Intégrales de Lebesgue, Fonctions d'ensemble, Classes de Baire.* 1ère éd. Paris, 1916, 2e éd. 1934.

Si le point P appartient au noyau, $1/PQ$ passe par l'infini et il faut une nouvelle définition.

Supposons d'abord que les masses soient positives. Nous assignons un nombre positif ρ et nous désignons par $[PQ]$ la longueur PQ bornée à ρ, c'est-à-dire que $[PQ]$ est égal à PQ si cette longueur est $> \rho$ et à ρ dans le cas contraire. Le potentiel $U(P)$ est la limite quand ρ tend vers zéro de l'intégrale (fonction continue de P)

$$\int \frac{d\mu(Q)}{[PQ]}.$$

Cette limite peut être finie ou infinie positive.

Si les masses sont des deux signes, leur potentiel est la différence des potentiels positifs et négatifs, dont l'un au moins doit avoir une valeur finie.

Le potentiel de masses positives est, par définition, limite de fonctions continues croissantes. C'est une fonction *semi-continue inférieurement*. Cette propriété se définit par la condition suivante : si le point P' tend vers P, on a

$$\lim U(P') \geqslant U(P) ;$$

c'est-à-dire qu'en un point de discontinuité, la fonction ne peut sauter que vers le haut.

Voici deux remarques élémentaires concernant les potentiels de masses positives. Elles sont dues à M. Vasilesco * et nous seront utiles dans la suite.

La borne du potentiel sur l'espace est au plus double de celle sur le noyau.† En effet, la distance d'un point M du noyau au point P extérieur est au moins la moitié de celle au point M_0 du noyau le plus rapproché de P.

En un point P du noyau l'oscillation du potentiel sur l'espace est au plus double de celle sur le noyau. On voit facilement que cette oscillation est la borne du potentiel dû aux masses contenues dans une sphère infiniment petite de centre P. Ainsi, ce théorème est la conséquence du précédent. En particulier, *en un point où le potentiel est continu sur le noyau, il l'est aussi sur l'espace.*

3. Loi de réciprocité.

Soient μ une distribution sur un ensemble E et U son potentiel, ensuite α une distribution sur un ensemble A et V son potentiel ; on a

$$\int_A U d\alpha = \int_E V d\mu.$$

C'est ce que nous appelons la *loi de réciprocité*. Elle est toujours exacte si les masses ne changent de signe ni sur A ni sur E. Si l'on

* *C. R. Ac. des Sc. Paris*, 200 (1935), p. 1173.

† En réalité, la borne du potentiel est toujours atteinte sur le noyau, mais la démonstration suppose la théorie du balayage comme l'a démontré M. A. J. Maria. Voir Frostman, No. 38, p. 68.

exprime U et V en intégrales, on obtient, à l'ordre d'intégration près, la même intégrale double dans les deux membres, et l'inversion se justifie aisément. Si les masses α et μ sont de signes variables, les intégrales doubles doivent être déterminées.

Voici une première application de cette formule.

Soit Ω le domaine intérieur à une surface sphérique S de rayon R et de centre P. Définissons la distribution α en plaçant la masse 1 en P et en distribuant uniformément la masse -1 sur S. Soit α_1 la même distribution uniforme de la masse $+1$. Le potentiel V de la distribution α est, en désignant par r la distance à P,

$$V(r)=\frac{1}{r}-\frac{1}{R} \text{ dans } \Omega, \quad V(r)=0 \text{ ailleurs.}$$

La formule devient donc

$$\int_P - \int_S U d\alpha_1 = \int_E V d\mu = \int_\Omega V d\mu ;$$

c'est-à-dire

$$U(P) - \int_S U d\alpha_1 = \int_\Omega V d\mu.$$

L'intégrale sur S est la moyenne de U sur S et V est positif. Donc la valeur de U au point P est égale, ou supérieure, ou inférieure à sa moyenne sur la sphère, selon que celle-ci ne renferme pas de masses, ou en renferme d'exclusivement positives, ou d'exclusivement négatives. Suivant qu'on est dans le second ou le troisième cas, U est *surharmonique* ou *sousharmonique*. En particulier, une fonction n'admet pas de maximé à l'intérieur d'un domaine où elle est sousharmonique.

La consideration de la sphère Ω conduit à une autre conséquence utile. Si la valeur de U, au centre P, ne surpasse aucune de celles à la surface S, la formule précédente donne

$$\int_\Omega V d\mu \leqslant 0.$$

On en conclut le théorème suivant que nous allons démontrer :

THÉORÈME. *Si le potentiel atteint au point P, intérieur à un domaine Ω, sa borne inférieure dans ce domaine, la dérivée inférieure de la distribution est nulle ou négative en ce point.*

La notion de dérivée d'une distribution μ se confond avec celle de densité. La dérivée de μ au point P est la limite du quotient

$$\frac{\mu(\omega)}{\text{mes. } \Omega}$$

de la masse contenue dans une sphère infiniment petite ω de centre P par la mesure de cette sphère. Si la limite est inexistante, les plus grande et plus petite limites sont les dérivées supérieure et inférieure de la distribution. Passons à la démonstration du théorème.

Dans l'hypothèse du théorème, la dernière relation a lieu pour

toute sphère Ω centrée sur P et de rayon R suffisamment petit. Désignons par $V(r)$ la valeur constante de V à la distance r de P, par $\phi(r)$ la masse contenue dans la sphère (ouverte) ω de rayon r ; la dite relation prend la forme

$$\int_\Omega V d\mu = \int_0^R V(r) d\phi(r) \leqslant 0.$$

Supposons d'abord cette intégrale négative ; je dis que $\phi(r)$ passe par des valeurs négatives sous le signe $\int$. Le facteur V décroît avec r et est infini pour $r=0$; remplaçons-le par un facteur *borné* v, égal à V si $V>L$ et à L dans le cas contraire ; l'intégrale

$$\int_0^R v d\phi$$

tend vers la précédente quand L croît indéfiniment et devient donc négative pour L assez grand. Décomposons l'intervalle $(0, R)$ par des points intermédiaires $0, r_1, r_2, \ldots r_{n-1}, r_n = R$ et marquons du même indice les valeurs de v et de ϕ en chacun de ces points. L'intégrale précédente est la limite de la somme

$$\begin{aligned} &v_1\phi_1 + v_2(\phi_2 - \phi_1) + \ldots + v_n(\phi_n - \phi_{n-1}) \\ &\quad = (v_1 - v_2)\phi_1 + (v_2 - v_3)\phi_3 + \ldots + (v_{n-1} - v_n)\phi_{n-1} + v_n\phi_n, \end{aligned}$$

quand les intervalles (r_k, r_{k+1}) tendent vers zéro. La somme en seconde ligne devient donc négative ; dès lors il y a des ϕ négatifs, sinon (v étant décroissant) tous les termes seraient positifs. Cette conclusion s'applique quelque petit que soit R ; donc $\phi(r)$ prend des valeurs négatives pour des valeurs infiniment petites de r ; ou, ce qui est la même chose, $\mu(\omega)$ est négatif pour des sphères ω de rayon r infiniment petit, et la dérivée inférieure de μ est négative ou nulle.

Supposons, en second lieu, que l'intégrale soit constamment nulle quand r tend vers zéro, quelque petit que soit ϵ positif, l'intégrale

$$\int_0^R V(r) d[\phi(r) - \epsilon r^4]$$

sera négative ; donc $\phi(r) - \epsilon r^4$ prend des valeurs négatives sur des sphères ω de rayon r aussi petit qu'on veut ; auquel cas, on a

$$\frac{\phi}{r^3} < \epsilon r, \text{ d'où } \frac{\mu(\omega)}{\text{mes. } \omega} < \frac{3\pi}{4}\epsilon r.$$

Cette expression tend vers zéro avec le rayon r de la sphère ; donc la dérivée inférieure de μ est encore nulle ou négative.

Le théorème est démontré. On prouve dans la théorie des fonctions d'ensemble qu'une telle fonction ne peut être positive sur un ensemble en chaque point duquel sa dérivée est nulle ou négative. Donc le théorème précédent entraîne le suivant :

Un ensemble, en chaque point duquel le potentiel atteint sa borne inférieure dans un domaine, ne peut porter de charge positive.

De même (par un changement de signe) *un ensemble de points en chacun desquels le potentiel atteint sa borne supérieure dans un domaine, ne peut porter de charge négative.*

En particulier, *il ne peut y avoir de masses à l'intérieur d'un domaine où le potentiel est constant.*

Voici concernant ces derniers théorèmes une remarque importante pour la suite. Nous définirons bientôt ce qu'il faut entendre par *ensemble de capacité nulle.* Nous verrons que, sous certaines conditions, ils peuvent être négligés dans l'intégration par rapport aux masses, tout comme les ensembles de mesure nulle dans les intégrales de Lebesgue. C'est, en particulier, le cas pour l'intégrale

$$\int_\Omega V d\mu$$

sur laquelle reposent les démonstrations précédentes. Les trois derniers théorèmes subsistent donc quand, pour la détermination des bornes supérieure ou inférieure du potentiel dans un domaine, on fait abstraction dans ce domaine d'un ensemble de points de capacité nulle, et quand on néglige un sousensemble de capacité nulle dans l'ensemble considéré.

4. Limites de Distributions.*

Considérons une suite de distributions positives $\mu_1, \mu_2, \ldots \mu_n, \ldots$ dans un domaine D. On suppose que la frontière de ce domaine ne porte aucune masse. Nous disons que la distribution μ sur D est la *limite* de la suite des distributions précédentes si $\mu(e)$ est la limite de $\mu_n(e)$ sur tout sousensemble e de D dont la frontière ne porte aucune masse de la distribution μ. Il peut arriver que les distributions $\mu_1, \mu_2, \ldots$ définissent les positions successives de masses que l'on peut suivre individuellement dans leurs déplacements, la suite converge si ces masses tendent vers des positions limites.

A cette définition se rattachent quelques théorèmes fondamentaux concernant le passage à la limite sous le signe $\int$.

1°. *Théorème généralisé de Lebesgue.* Soit $F_1, F_2, \ldots F_n, \ldots$ une suite de fonctions de point mesurables et bornées dans leur ensemble ; si elles ont une limite F, on a

$$\lim \int F_n d\mu = \int F d\mu.$$

2°. Si la suite de distributions positives $\mu_1, \mu_2, \ldots \mu_n, \ldots$ converge vers la distribution μ et si la fonction de point F est continue, on a

$$\lim \int F d\mu_n = \int F d\mu.$$

* Cette théorie et son application à l'intégration a été exposé dans notre Mémoire cité des *Annales de l'Institut Poincaré* (1932). Elle l'est aussi dans l'ouvrage de Frostman. Nous sommes revenus en détail sur ces questions dans nos conférences de Strasbourg et nous y renvoyons le lecteur.

3°. Supposons que la suite de distributions positives $\mu_1, \mu_2, \ldots \mu_n, \ldots$ converge vers la distribution μ ; soient alors $U_1, U_2, \ldots U_n, \ldots$ et U les potentiels correspondants, on a, l'intégration s'étendant à toute la masse,

$$U \leqslant \lim U_n, \int U d\mu \leqslant \lim \int U_n d\mu_n.$$

Mais si l'on a $U \geqslant U_n$, l'égalité a lieu (U supposé borné).

Nous ne démontrerons ici que les dernières inégalités. Soit $[PQ]$ la valeur de PQ limitée inférieurement à ρ. On a (par 2°)

$$U_n > \int \frac{d\mu_n(Q)}{[PQ]}, \ \lim U_n > \int \frac{d\mu(Q)}{[PQ]};$$

et en faisant tendre ρ vers zéro, on obtient $\lim U_n \geqslant U$. La seconde inégalité s'obtient par le même raisonnement en exprimant $\int U d\mu$ sous forme d'intégrale double.

Si l'on a $U \geqslant U_n$, il vient, en utilisant la loi de réciprocité,

$$\lim \int U_n d\mu_n < \int U d\mu_n = \int U_n d\mu ;$$

et, à la limite, pour $n = \infty$, on obtient (par 1°)

$$\lim \int U_n d\mu_n \leqslant \int U d\mu.$$

Nous avons obtenu deux inégalités de sens différents, il y a donc égalité. Les inégalités du théorème 3° constituent, pour les distributions positives, ce que nous appelons le *principe d'abaissement*.

5. Capacité d'un ensemble.*

La définition de la capacité d'un ensemble de points est une extension naturelle de celle que l'on donne en physique de la capacité d'un corps conducteur.

La capacité d'un ensemble est la borne supérieure des charges positives qu'il peut soutenir sans que son potentiel surpasse l'unité.

Cette borne peut être accessible ou non. Elle est accessible pour un domaine régulier fermé. On sait, en effet, depuis Gauss, qu'il existe une répartition d'équilibre qui rend le potentiel égal à 1 sur tout le domaine. La charge ainsi répartie s'appellera la charge capacitaire, et son potentiel le potentiel capacitaire. Par contre, la charge limite est inaccessible pour le même domaine ouvert, car la limite ne serait atteinte que par une charge portée par la frontière et que celle-ci est exclue du domaine ouvert.

Il est immédiat que la capacité d'un sous-ensemble ne peut surpasser celle de l'ensemble entier.

D'autre part, la capacité d'un ensemble E est la borne supérieure de celles de ses sous-ensembles et, plus particulièrement de celles de

* Nous avons donné cette définition dans notre Mémoire cité de l'Institut Poincaré. M. Wiener en avait donné antérieurement une définition qui ne pouvait s'appliquer qu'aux ensembles fermés.

ses sous-ensembles fermés. En effet, d'après la théorie des fonctions d'ensemble, si l'on donne à E une charge λ aussi voisine qu'on veut de sa capacité, une part aussi grande qu'on veut de cette charge est portée par un sous-ensemble convenable de E, et, en particulier, par un sous-ensemble fermé.

Enfin la capacité d'un ensemble fermé peut être rattachée à celles d'une suite de domaines réguliers. En effet, tout ensemble fermé F peut être considéré comme la partie commune à une suite de tels domaines emboîtés $D_1 > D_2 > \ldots > D_n > \ldots$, alors la capacité de F est la limite de la suite décroissante des capacités des domaines précédents. En effet, la capacité de F ne peut être supérieure à cette limite, de sorte que si nous désignons par $\mu_1, \mu_2, \ldots \mu_n, \ldots$ les distributions capacitaires des domaines précédents, suite que l'on peut supposer converger vers une distribution limite μ (en vertu du théorème du choix), la capacité de F ne peut surpasser $\mu(F)$. Elle n'est pas moindre non plus, en vertu du principe d'abaissement, car en désignant les potentiels par la lettre U avec et sans indice, on a $U \leqslant \lim U_n \leqslant 1$.

6. Ensemble de capacité nulle.

Comme cas particulier de la définition de la capacité, *un ensemble de capacité nulle est un ensemble auquel toute charge positive, si petite qu'elle soit, donne un potentiel non borné.*

D'après la remarque faite en dernier lieu, un ensemble fermé de capacité nulle peut être enfermé dans un domaine régulier de capacité aussi petite que l'on veut.

Dans l'intégration par rapport aux masses, les ensembles de capacité nulle jouissent d'un privilège important tant que les potentiels de ces masses sont bornés ; ils peuvent être négligés dans l'intégration comme ceux de mesure nulle dans les intégrales de Lebesgue. En effet, une intégrale de Stieltjes sur un ensemble E est la somme des intégrales sur chaque partie de E et l'intégrale est nulle sur un ensemble de capacité nulle, lequel ne peut soutenir une masse de potentiel borné. C'est cette propriété que nous avons invoquée pour généraliser les théorèmes énoncés précédemment comme application de la loi de réciprocité (3).

Comme nous le verrons, l'ensemble de capacité nulle est la véritable clé pour résoudre le problème généralisé de Dirichlet. Cette notion se rencontre déjà au principe même des théorèmes d'unicité.

7. Théorème d'unicité des fonctions harmoniques.

Si une fonction harmonique $U(P)$ est bornée dans un domaine ouvert Ω, et si elle tend vers zéro quand le point P tend vers un point de la frontière F, à l'exception d'un ensemble de points frontières de capacité nulle, on a $U = 0$ dans tout Ω.

Appelons *oscillation* de U en un point frontière exceptionnel la différence de ses deux limites d'indétermination. Cette oscillation est bornée par hypothèse. L'ensemble ε des points où l'oscillation est $\geqslant \epsilon$ est fermé et de capacité nulle. Dès lors, on peut enfermer ε

dans un domaine régulier Δ, de capacité suffisamment petite pour qu'une masse λ, aussi petite qu'on voudra, admette un potentiel d'équilibre u aussi grand qu'on veut sur Δ, donc supérieur aux oscillations de U. Ceci fait, la fonction harmonique $U+u+\epsilon$ est positive sur toute la frontière de $\Omega-\Delta$, tandis que $U-u-\epsilon$ y est négative ; donc la première fonction est positive dans tout $\Omega-\Delta$ et la seconde négative. Faisons tendre ϵ, Δ et λ vers zéro ; u tend vers zéro en tout point intérieur à Ω, et l'on a, par conséquent, $U=0$ dans tout Ω.

Il suit de ce théorème que deux fonctions harmoniques, bornées dans un domaine ouvert Ω, et qui tendent vers la même limite sur le bord à l'exception d'un ensemble de points frontières de capacité nulle, sont identiques dans Ω.

Voici le théorème qui correspond au précédent concernant les fonctions sousharmoniques :

Si dans l'énoncé du théorème précédent, on remplace le mot harmonique *par* sousharmonique, *on a* $U<0$ *dans tout* Ω.

Définissons u comme dans la démonstration précédente, de façon que $U-u-\epsilon$ soit partout négatif sur le bord de Ω. La fonction sousharmonique $U-u-\epsilon$ sera negative dans Ω, sinon elle aurait dans Ω une borne supérieure nulle ou positive et elle l'atteindrait, car elle ne peut sauter vers le haut. Mais une fonction sousharmonique dans un domaine ne peut atteindre son maximum à l'intérieur et il y a contradiction. Faisons tendre u vers zéro dans Ω; nous avons, à la limite, $U\leqslant 0$. Mais l'égalité est impossible, car U atteindrait son maximum à l'intérieur de Ω et nous venons de dire que c'est impossible. On a donc $U<0$ dans tout Ω.

8. THÉORÈME.

Le potentiel d'une masse positive finie ne peut être infini que sur un ensemble de capacité nulle.

Soit μ la distribution de cette masse sur un ensemble E et U son potentiel. Si U était infini sur un ensemble e de mesure non nulle, on pourrait répartir sur e une masse positive h de potentiel borné H. On aurait alors, par la loi de réciprocité,

$$\int_e U dh = \int_E H d\mu,$$

ce qui est impossible, car le premier membre est infini et le second fini.

Ce théorème, démontré par M. Frostman, s'applique encore à des potentiels généralisés.*

9. NOYAU FERMÉ ET NOYAU RESTREINT.†

On démontre, dans la théorie des fonctions d'ensemble, qu'une fonction positive d'ensemble μ prend toutes ses valeurs non nulles

* Frostman, ouvrage cité, No. 44, p. 81.

† Ces termes sont nouveaux. Les propriétés invoquées des ensembles sont établies dans mon ouvrage cité de la Collection Borel.

sur l'ensemble des points où la dérivée μ' est positive. Cet ensemble porte donc toute la masse quand μ est une distribution positive, mais il n'est généralement pas fermé. C'est un sous-ensemble du noyau de masse considéré précédemment. Nous donnerons le nom de *noyau restreint* à tout ensemble analogue qui porte toute la masse. Il a pour ensemble dérivé le noyau proprement dit que nous appellerons, par opposition, *le noyau fermé*.

Si l'on considère une distribution de masses des deux signes, on démontre dans la théorie des fonctions d'ensemble qu'elle est la différence $\lambda - \mu$ de deux distributions positives admettant respectivement deux noyaux restreints sans point commun. La distribution λ admet comme noyau restreint l'ensemble des points où la distribution proposée admet une dérivée positive, la distribution μ celui des points où la dérivée est négative. Les deux noyaux fermés correspondants peuvent avoir des points communs et même coïncider.

Le potentiel d'une distribution positive admet la même borne supérieure sur le noyau fermé que sur un noyau restreint, car tout point du premier est limite de points du second, et le potentiel (continu inférieurement) ne peut faire de saut vers le bas. Sa borne sur l'espace est au plus double de celle sur le noyau, car la comparaison des distances aux masses prouve que le potentiel en un point extérieur au noyau ne peut surpasser le double du potentiel au point le plus rapproché du noyau fermé.

10. Minimum d'une intégrale définie.

Deux des problèmes les plus importants de la théorie du potentiel sont liés à la condition de minimer une certaine intégrale définie, déjà considéré par Gauss * il y a près d'un siècle. Ce sont les problèmes du signe de l'intégrale d'énergie et du balayage.

Soit α une distribution donnée de masses positives sur un ensemble A fermé ou non. Nous supposons que le potentiel de cette distribution est simplement borné. Contrairement à ce qui a été fait jusqu'à présent, nous ne le supposons pas continu. Soit ensuite E un second ensemble, celui-ci fermé. Les deux ensembles A et E peuvent empiéter ou non. On demande, parmi toutes les distributions positives μ de potentiel U que l'on peut répartir sur E, celle qui minime l'intégrale

$$g(\mu) = \int_E (U - 2V)\, d\mu.$$

Nous avons d'abord à prouver l'existence d'un minimum, ce que Gauss pensait inutile. Pour cela nous avons une remarque préliminaire à faire.

Si dans l'intégrale précédente sur E, on a $U - 2V > 0$ sur un sous-

* C. F. Gauss, Allgemeine Lehrsätze in Beziehung auf die im verkehrten Verhältnisse des Quadrats der Entfernung wirkenden Anziehungs- und Abstossungs-Kräfte, *Werke* 5, p. 232.

ensemble e de E et que e porte des masses μ, on diminue $g(\mu)$ en supprimant ces masses. On a, en effet,

$$g(\mu)=\int_{E-e}(U-2V)d\mu+\int_{e}(U-2V)d\mu.$$

Si l'on supprime les masses μ portées par e, on diminue U dans la première intégrale, et l'on supprime la seconde qui est positive, donc $g(\mu)$ décroît. Cela fait, U désignant le nouveau potentiel, on a $U-2V\leqslant 0$ sur un noyau restreint de la distribution réduite et la borne de U est au plus quadruple de celle de V sur tout l'espace (2).

Revenons à l'existence d'un minimum. L'intégrale $g(\mu)$ est négative si les masses μ sont suffisamment réduites pour que l'on ait $U<2V$, et elle devient infinie positive avec les masses μ ; donc elle admet une borne inférieure $-L$. Nous allons montrer que cette borne est accessible.

Assignons une suite de distributions $\mu_1, \mu_2, \dots \mu_n, \dots$ de potentiels $U_1, U_2, \dots U_n, \dots$ telle que $g(\mu)$ tende vers sa borne $-L$.*. Ces potentiels peuvent être supposés bornés dans leur ensemble, en vertu de la remarque préliminaire que nous venons de faire. De plus, en vertu du théorème du choix, on peut supposer la suite des distributions convergente et tendant vers une distribution limite μ, dont le potentiel U sera borné aussi, car il est $\leqslant \lim U_n$ en vertu du principe d'abaissement. Je dis que cette distribution μ minimise $g(\mu)$. On a, en effet, par le principe d'abaissement

$$\lim\int_E U_n d\mu_n \geqslant \int_E U d\mu\ ;$$

puis, par la loi de réciprocité et le théorème généralisé de Lebesgue,

$$\lim\int V d\mu_n=\lim\int U_n d\alpha=\int U d\alpha=\int V d\mu.$$

Donc

$$-L=\lim\int_E(U_n-2V)d\mu_n\geqslant\int_E(U-2V)d\mu.$$

Mais l'égalité seule est possible, puisque $-L$ est, par hypothèse, la borne inférieure de la dernière intégrale.

Voici maintenant quelques propriétés de la distribution minimisante μ.

1°. *A l'exception au plus d'un ensemble de points de capacité nulle, on a, sur tout l'ensemble E,*

$$U-V\geqslant 0.$$

Si $U-V$ était négatif sur un sous-ensemble e de E de capacité positive, on pourrait faire sur e une distribution positive h de potentiel borné H. Soit ϵ un coefficient positif ; la distribution

* Il y a ici un usage, tout au moins apparent, de l'axiome de Zermelo. J'ai suivi dans mes conférences de Strasbourg une marche beaucoup plus longue qui évite cette objection.

positive $\mu + \epsilon h$ de potentiel borné $U + \epsilon H$ donnerait à $g(\mu)$ la variation

$$\delta g = g(\mu + \epsilon h) - g(\mu).$$

On aurait, en utilisant la loi de réciprocité,

$$\delta g = 2\epsilon \int_e (U - V)\,dh + \epsilon^2 \int_e H dh.$$

Le premier terme au second membre serait négatif et prépondérant pour ϵ assez petit. Donc g diminuerait, ce qui est contre l'hypothèse.

2°. *La distribution μ admet un noyau restreint sur lequel on a partout $U - V = 0$.*

Montrons d'abord qu'un ensemble e sur lequel on a partout $U - V > 0$ ne peut porter de masse positive. Soit, par impossible, h la distribution de cette masse sur e, et H son potentiel (qui est borné). En revenant à l'expression précédente de δg, on voit que l'on peut prendre h assez petit pour que la distribution encore positive $\mu - \epsilon h$ rende δg négatif, ce qui est impossible. Donc la distribution μ admet un noyau restreint contenu dans l'ensemble des points où $U - V \leqslant 0$, et comme l'ensemble des points où $U - V < 0$ est de capacité nulle, on peut encore le retrancher de ce noyau.

3°. *Quand les ensembles A et E sont empiétants, les masses μ et α peuvent se détruire partiellement ; la distribution μ (ou, s'il y a réduction, sa portion non détruite μ') est sur la frontière de E.*

En effet, soit P un point du noyau restreint considéré ci-dessus de la distribution μ. Si ce point est intérieur à E, on a, en ce point, $U = V$ et au voisinage $U \geqslant V$ (abstraction faite d'un ensemble inopérant de capacité nulle). Donc $U - V$ est minimé au point P, et l'ensemble de ces points ne peut porter de masse positive (3).

11. INTÉGRALE D'ÉNERGIE.

Soit μ une distribution de masses des deux signes sur un ensemble fermé et U son potentiel ; *l'intégrale, dite intégrale d'énergie*

$$I = \int_E U d\mu$$

*est positive et ne s'annule qu'avec les masses.**

La distribution μ est la différence $\lambda - \alpha$ de deux distributions positives de potentiels U_1 et V. L'intégrale d'énergie devient

$$I = \int_E (U_1 - V)(d\lambda - d\alpha) = \int_E (U_1 - 2V)\,d\lambda + \int_E V d\alpha.$$

Supposons d'abord les potentiels bornés. Enfermons E à l'intérieur d'une surface sphérique S ayant son centre sur E et de rayon croissant indéfiniment. Etendons l'intégrale précédente à tout

* C'est la première fois, pensons-nous, que la démonstration de ce théorème est rattachée au minimum de l'intégrale $g(\mu)$. Gauss suppose l'existence et la continuité des dérivées partielles de U et transforme I en une intégrale étendue à tout l'espace et portant sur la somme des carrés des trois dérivées premières de U. On doit à M. M. Riesz une transformation de I en une intégrale portant encore sur un carré. Elle est très ingénieuse, mais fort délicate. Voir Frostman, No. 16, p. 28.

l'intérieur Ω de la sphère et déterminons son minimum pour toutes les distributions positives λ possibles dans Ω, mais pour la même distribution α. Nous obtenons ainsi une borne inférieure de I, car toute distribution sur E en est une sur Ω. La dernière intégrale $\int V d\alpha$ étant fixe, il faut minimer l'intégrale

$$\int_\Omega (U_1 - 2V)\, d\lambda.$$

La distribution $\lambda - \alpha$ sur Ω se réduit à une distribution $\lambda' - \alpha'$ de potentiel $u - v$, dont les noyaux restreints n'empiètent pas. Si elle minimise l'intégrale, le noyau de λ' est sur la frontière S et l'on a sur ce noyau restreint $u = v$. On a donc

$$I = \int_\Omega (u - v)(d\lambda' - d\alpha') = -\int_E (u - v)\, d\alpha'.$$

Je dis que cette expression tend vers zéro quand la surface S s'écarte à l'infini. En effet, le maximum v_0 de v sur S tend vers zéro sur S, et u qui est $< v_0$ sur S (et $< 2v_0$ sur l'espace) tend vers zéro sur E ; donc v, qui est $\leqslant u$ sur E, tend aussi vers zéro, et la dernière intégrale tend vers zéro avec $u - v$. En définitive, l'intégrale I, admettant la borne inférieure zéro, est positive ou nulle.

Si les potentiels ne sont pas bornés, on arrive à la même conclusion par un passage à la limite. On remplace λ et α par les distributions à potentiels bornés qu'on obtient en supprimant les masses sur l'ensemble des points où leur potentiel surpasse une borne assignée L. Les nouvelles distributions tendent vers les premières quand L tend vers l'infini, et leurs potentiels (par définition) tendent vers les potentiels primitifs. L'intégrale modifiée est nulle ou positive, donc sa limite l'est aussi.

L'intégrale d'énergie ne s'annule qu'avec les masses.

En effet, si les masses ne disparaissent pas, leur potentiel est de signe déterminé, par exemple positif, sur un ensemble e de capacité positive. En plaçant sur e une masse négative suffisamment petite, on fera baisser (comme on l'a fait précédemment) la valeur de l'intégrale sans qu'elle puisse devenir négative ; donc elle était positive.

12. Le Balayage.

Revenons aux conditions de minimum de l'intégrale définie

$$g(\mu) = \int_E (U - 2V)\, d\mu,$$

dans laquelle V est le potentiel d'une distribution positive α donnée sur un ensemble A, et U celui d'une distribution μ à choisir sur un ensemble E donné et, celui-ci, fermé. Nous ajouterons aux hypothèses antérieures deux conditions particulières :

1°. Nous supposons que l'ensemble A n'empiète pas sur E ; il ne se fait alors aucune réduction entre les distributions μ et $-\alpha$.

2°. Nous supposons que V est continu en tout point de E, de

sorte que $U - V$ jouit sur E des propriétés de la semi-continuité inférieure comme un potentiel positif.

Nous savons déjà que dans le cas du minimum, la distribution μ est portée par la frontière de E et qu'elle admet un noyau restreint sur lequel $U - V = 0$. Il s'ensuit que $U - V$ est harmonique dans tout domaine intérieur à E. Nous savons encore que $U - V \geqslant 0$ sur tout E, à l'exception d'un ensemble de capacité nulle, et cet ensemble exceptionnel, dans le cas présent, ne peut plus se trouver que sur la frontière de E.

Soit N le noyau fermé de la distribution μ. Il fait partie de la frontière de E et est le dérivé du noyau restreint. En vertu de la semi-continuité de $U - V$, qui s'annule sur le noyau restreint, on a $U - V \leqslant 0$ sur le noyau fermé, et l'inégalité ne peut exister que sur un ensemble de capacité nulle. Comme $U - V$ ne peut faire de saut vers le bas, il suit de là que $U - V$ est continu sur le noyau en tout point du noyau où $U - V = 0$.

C'est maintenant un théorème élémentaire connu (2) que si un potentiel de masses positives U est, en un point du noyau, continu sur celui-ci, il l'est aussi sur l'espace. Donc, en tout point du noyau N, sauf un ensemble de capacité nulle où $U - V < 0$, cette fonction est continue sur l'espace.

Le noyau fermé N est la frontière de son ensemble complémentaire sur l'espace. Cet ensemble complémentaire de N est ouvert. Il est connexe ou composé de régions connexes $\Omega_1, \Omega_2, \ldots$. Dans une de ces régions Ω contenant des masses α, $U - V$ est sousharmonique et s'annule sur le bord (exception faite d'un ensemble de capacité nulle), on a donc $U - V < 0$, et cette région ne peut comprendre de points de E. Si E admet des points intérieurs, il y a donc d'autres régions Ω. Dans une région Ω dépourvue de masses, $U - V$ se comporte sur le bord comme dans le cas précédent, mais est harmonique ; on a donc $U - V = 0$. Une frontière de E qui ne borde pas une région Ω contenant des masses α, ne peut porter de masses μ, puisque $U - V$ y est minimé.

Il suit de là que le noyau N fait la coupure entre la ou les régions connexes Ω qui contiennent des masses et le reste de l'espace ; et qu'il se confond, par conséquent, avec les frontières de E qui limitent la ou les régions connexes contenant des masses α.

Nous pouvons énoncer le théorème suivant :

La condition nécessaire et suffisante pour que la distribution μ soit, parmi toutes les distributions possibles de signe quelconque (bornées ou non), celle qui minimise l'intégrale $g(\mu)$, est que $U - V$ soit nul sur E, à l'exception au plus d'un ensemble de capacité nulle. De plus, cette distribution minimisante est unique.

Soit μ une distribution vérifiant cette condition et $\mu + h$, de potentiel $U + H$, une autre distribution quelconque sur E. L'intégrale de $(U - V)\,dh$ est nulle et il vient

$$g(\mu + h) - g(\mu) = \int_E H\,dh.$$

La dernière intégrale est positive sauf pour $h=0$. Donc la distribution μ est minimisante et elle est la seule.

L'opération du balayage consiste à remplacer une distribution de masses α extérieures à E par une distribution sur la frontière qui maintient le même potentiel sur E, à l'exception d'un ensemble de points-frontières de capacité nulle. Nous venons de voir que la solution est unique.

Nous avons considéré jusqu'ici le balayage de masses positives, celui de masses négatives revient au précédent par un simple changement de signe de toutes les masses. Celui de masses α des deux signes comporte le balayage séparé des masses positives et négatives. On constate que l'on obtient ainsi la distribution μ de signe quelconque qui minime l'intégrale $g(\mu)$. La démonstration est la même que pour les masses positives et la solution est unique.

13. Balayage de masses positives dans le cas où V est discontinu (mais borné) sur E.

Ce cas ne se présente que si les masses α touchent à l'ensemble E, d'où elles sont supposées exclues. Dans ce cas α se décompose dans la somme d'une infinité dénombrable de distributions $\alpha_1, \alpha_2, \ldots \alpha_n, \ldots$ de potentiels $v_1, v_2, \ldots v_n, \ldots$ dont la somme est égale à V. Le balayage de α_n porte sur E une distribution μ_n de potentiel u_n. On a, sur E, $u_n - v_n = 0$, exception faite d'un ensemble de capacité nulle. Hors de E, on a $u_n - v_n = 0$ dans les domaines connexes (bornés par E) dépourvus de masses α, et $u_n - v_n < 0$ dans ceux qui contiennent des masses α_n.

Soit μ la somme des μ_n et U son potentiel somme des u_n. On a

$$U - V = \Sigma(u_n - v_n).$$

La somme d'une infinité dénombrable d'ensembles de capacité nulle est de capacité nulle. Donc, sur E, on a $U - V = 0$ abstraction faite d'un ensemble de capacité nulle. Hors de E on a $U - V \leqslant 0$ selon que l'on se trouve dans un domaine connexe (borné par E) dépourvu ou pourvu de masses α.

Il suit du comportement de $U - V$ sur E que l'on a, pour toute distribution h,

$$\int_E (U - V)\,dh = 0,$$

ce qui est la condition pour que la distribution μ minimise et minimise seule l'intégrale définie

$$\int_E (U - 2V)\,d\mu.$$

Dans ce cas, le balayage jouit donc des mêmes propriétés que dans l'hypothèse où V est continu sur E. La seule différence est que $U - V$ peut n'être pas continu aux points du noyau de la distribution μ où $U - V = 0$.

14. BALAYAGE PROGRESSIF.

Le balayage sur un ensemble fermé E peut aussi se faire progressivement. Si l'ensemble E est commun à une suite d'ensembles emboîtés $E_1 > E_2 > \ldots > E_n > \ldots$ en nombre fini ou infini. On peut effectuer successivement le balayage sur E_1 ; puis, tenu compte de celui-ci, le balayage sur E_2, et ainsi de suite. S'il n'y a qu'un nombre limité d'ensembles E_n, le dernier balayage se fait sur E et donne le même résultat que le balayage direct, puisqu'il n'y a rien de changé à l'intégrale à minimer en dernier lieu. S'il y a une infinité d'ensembles E_n, on suppose que E est leur ensemble limite commun, alors la distribution μ du balayage sur E et son potentiel U sont limites de la distribution μ_n sur E_n et de son potentiel U_n. Nous pouvons supposer la suite des distributions μ_n convergente vers une distribution μ sur E de potentiel U, car notre raisonnement s'appliquera à toute suite convergente qu'on pourrait en extraire. Je dis que μ et U sont la distribution et le potentiel résultant du balayage, parce qu'elles minimisent l'intégrale

$$g(\mu) = \int_E (U - 2V)\, d\mu.$$

Désignons, en effet, par V_n le potentiel des masses à balayer sur E_n ; quelle que soit la distribution h, nous avons, puisque $U_n - V_n$ s'annule sur E_n, donc sur E, sauf exception négligeable,

$$\int_E (U_n - V_n)\, dh = 0 ;$$

et, à la limite, par le théorème généralisé de Lebesgue, et en nous bornant toujours au cas où le potentiel V est borné sur E,

$$\int_E (U - V)\, dh = 0,$$

ce qui est la condition de minimum de $g(\mu)$.

Dans l'application de ce procédé, on suppose U_n borné sur l'ensemble E_n de même indice.

15. DISTRIBUTION D'ÉQUILIBRE. DISTRIBUTION ET POTENTIEL CAPACITAIRE.

Si les masses α sont uniformément réparties sur une surface sphérique S enfermant l'ensemble E, son potentiel est une constante M sur E. Le balayage sur E détermine une répartition de masses μ dont le potentiel est égal à M sur E, sauf pour un ensemble de points frontières de capacité nulle, et est < 1 dans la portion infinie de l'espace extérieure à E. Les points frontières exceptionnels se trouvent exclusivement sur la frontière de cette région connexe infinie, c'est-à-dire sur la frontière extérieure de E.

Si $M = 1$, nous obtenons la *distribution capacitaire* sur E, son potentiel est le *potentiel capacitaire*. Il est égal à 1 sur tout E et ses cavités, abstraction faite de points exceptionnels sur la frontière extérieure où il est < 1, et il est < 1 dans la portion infinie d'espace

extérieure à E. Nous conviendrons de désigner par λ et W la distribution et le potentiel capacitaire.

16. Calcul de la masse répartie sur E par le balayage.

Soit μ la distribution sur E provenant du balayage de masses α portées par un ensemble A. Soient U et V les potentiels des distributions μ et α. Soit λ et W la distribution capacitaire et son potentiel. Nous avons, par la loi de réciprocité et en observant que, sauf un ensemble négligeable, on a, sur E, $W=1$ et $U=V$,

$$\mu(E)=\int_E W d\mu=\int_E U d\lambda=\int_E V d\lambda=\int_A W d\alpha.$$

Si A est enfermé dans une cavité de E, on a $W=1$ sur A et, par conséquent, $\mu(E)=A(\alpha)$. Dans un balayage intérieur, toute la masse α est balayée sur E et il n'y a pas de perte de masses, comme si les masses étaient emprisonnées dans la cavité. Par contre, si A n'est pas enfermé, le balayage est extérieur, on a $W<1$, donc $\mu(E)<\alpha(A)$. Dans ce cas, il y a perte de masses et l'on peut concevoir qu'une partie des masses a été rejetée à l'infini.

17. Balayage polaire.

Supposons que l'unité de masse ait été concentrée en un point P ; nous dirons que le balayage de cette masse ponctuelle est un *balayage polaire* et que le point P est le *pôle du balayage*. Ce balayage porte sur E une distribution μ dont, r étant la distance à P, le potentiel U est égal a $1/r$ sur E abstraction faite d'un ensemble de capacité nulle. La masse $\mu(E)$ transportée sur E par le balayage est, par la formule précédente,

$$\mu(E)=W(P).$$

Donc, *dans un balayage polaire, la masse balayée sur E est égale à la valeur du potentiel capacitaire sur le pôle ; elle est égale à* 1 *pour un balayage intérieur et elle est* <1 *pour un balayage extérieur.*

La répartition μ sur E déterminée par un balayage polaire possède la propriété suivante, qui est fondamentale pour la résolution du problème de Dirichlet.

La charge $\mu(e)$ *portée par un sous-ensemble e de la frontière F de l'ensemble E est une fonction harmonique du pôle P.*

Voici la démonstration, telle que je l'ai donnée dans mon Mémoire de 1932. Le théorème se vérifie directement pour une sphère, car la densité de la couche μ s'exprime par une fonction harmonique du point P. Considérons maintenant un domaine ouvert Ω et sa frontière F. Enfermons le pôle P dans une sphère fixe S intérieure à Ω. Effectuons le balayage polaire en deux étapes, d'abord de P sur S, puis de S sur F. Soit λ la couche portée sur S par le premier balayage ; la charge $\lambda(d\sigma)$ sur un élément $d\sigma$ de S est harmonique ; son balayage sur un sousensemble e de F fournit une contribution proportionnelle à $\lambda(d\sigma)$, donc harmonique. Donc $\mu(e)$, somme des contributions analogues pour tous les éléments $d\sigma$, est harmonique.

18. POINTS IRRÉGULIERS.*

En vue du problème généralisé de Dirichlet, nous définirons d'abord les *points irréguliers* sur la frontière extérieure d'un ensemble fermé E, et nous en donnerons la définition suivante :

Un point irrégulier sur la frontière extérieure de E est un point de discontinuité de son potentiel capacitaire W.

Le potentiel W n'admet que des valeurs $\leqslant 1$ et, sur E, il n'est < 1 que sur un ensemble de capacité nulle. Cette fonction, étant semi-continue inférieurement, est continue en tout point où elle est égale à 1 (car elle ne peut sauter vers le bas). *Les points irréguliers sont donc ceux de la frontière où $W < 1$, et ils forment un ensemble de capacité nulle.*

Nous démontrons maintenant le théorème suivant :

Si un point P_0 de la frontière extérieure de E est irrégulier, il est possible de mener, dans le domaine Ω extérieur à E, une ligne continue L aboutissant à P_0 et telle que l'on ait, tout le long de cette ligne,

$$W(P) < W(P_0) < 1.$$

Au point irrégulier P_0 on a, par définition, $W(P_0) < 1$; mais $W(P_0)$ surpasse la valeur moyenne de $W(P)$ sur toute surface sphérique de centre P_0, car W est surharmonique. Or, sur E et éventuellement sur ses cavités, $W = 1$, à part un ensemble de points sans conséquence sur la valeur moyenne. Donc W est $< W(P_0)$ sur une portion de la surface sphérique comprise dans le domaine Ω extérieur à E. Si l'on fait tendre le rayon de la sphère vers zéro, un point choisi sur cette portion de la surface sphérique trace une ligne aboutissant à P_0.

Nous dirons qu'une ligne aboutissant au point irrégulier P_0 et le long de laquelle $W(P) < W(P_0)$, est une *ligne irrégulière*. Donc *tout point irrégulier admet une ligne irrégulière.*

Une ligne le long de laquelle $W(P)$ tend vers l'unité, est une *ligne régulière*. Le potentiel W est continu aux points réguliers où $W(P_0) = 1$. Donc un point régulier n'admet que des lignes régulières.

Rappelons-nous que $W(P)$ est la masse transportée sur E par un balayage de pôle P ; nous obtenons, dans le cas d'un balayage extérieur le théorème suivant :

Si le pôle P tend vers un point frontière P_0 le long d'une ligne régulière L, la masse unité tout entière est portée à la limite sur E. Au contraire, si la ligne est irrégulière, une perte finie de masses persiste à la limite.

Ce théorème se complète par le suivant :

Si la ligne L est régulière, la masse unité tout entière se concentre à la limite au point P_0 ; par contre, si la ligne est irrégulière, non seulement la masse balayée sur E reste déficiente à la limite, mais une part de cette masse ne se concentre pas au point P_0 et reste dispersée sur la frontière de E.

Considérons une sphère fixe, de centre P_0 et de rayon R arbitraire, contenant le pôle P. Le balayage polaire sur E peut se faire en

* Nous reproduisons ici les définitions et les démonstrations données dans nos Conférences de Strasbourg.

deux étapes : d'abord dans la portion du domaine Ω extérieur à E qui est borné par la surface sphérique, puis dans le domaine Ω entier. La perte de masse se produit dans le second balayage seul, et provient seulement de la masse portée par la portion σ de la surface sphérique intérieure à Ω. Ainsi, la ligne L est régulière ou non, suivant que les masses portées par σ après le premier balayage, tendent ou non vers zéro quand P tend vers P_0. Il faut, en effet, observer, que le second balayage dissipera nécessairement une partie finie des masses portées par σ si elles ne tendent pas vers zéro, car ces masses ne peuvent s'accumuler sur la lisière de E. Dans ce cas, la densité de la couche croîtrait indéfiniment, ce qui est impossible, car cette densité reste inférieure à celle que produirait le balayage polaire complet de la sphère. Si la ligne L est régulière, la masse entière se concentre en P_0 à la limite, puisqu'elle est enfermée sur E à l'intérieur d'une sphère de rayon R aussi petit qu'on veut. Si la ligne est irrégulière, une partie de la masse portée par σ est répartie par le second balayage sur toute la frontière de E.

Nous voyons, en même temps, que *la régularité* est une *propriété locale*, car elle dépend uniquement du comportement des masses à l'intérieur d'une sphère arbitrairement petite.

Ce caractère local permet d'étendre à une frontière interne la définition de la régularité ou de l'irrégularité d'un point frontière. Nous dirons qu'un point irrégulier sur la frontière d'une cavité est un point qui devient irrégulier quand on met la cavité en communication avec l'extérieur par un canal. Sauf que le balayage se fait sans perte, les autres propriétés du point irrégulier subsistent. À la limite, quand le point P tend vers P_0 par une ligne irrégulière, la masse balayée ne se concentre pas tout entière en P_0. Une partie plus ou moins grande, qui dépend du tracé de la ligne, reste dispersée sur toute la frontière de la cavité.

19. Problème généralisé de Dirichlet.

Soit Ω un ensemble ouvert connexe, d'un ordre de connexion quelconque et pouvant être infini, mais dont la frontière E est bornée. Soit donnée sur E une fonction U continue sur cet ensemble. Il existe une fonction $U(P)$ harmonique dans Ω et qui tend vers $U(Q)$ quand le point P tend vers un point Q de la frontière, à l'exception d'un ensemble de points frontières de capacité nulle. Cette fonction est régulière à l'infini si Ω est illimité. Le problème n'admet qu'une seule solution bornée. Celle-ci s'exprime par l'intégrale généralisée de Stieltjes

$$U(P) = \int_E U(Q)\, d\mu(Q),$$

où μ est la distribution sur E résultant du balayage polaire de pôle P.

En effet, $U(P)$ est harmonique avec μ, et tend vers $U(Q)$ quand P tend vers Q le long d'une ligne régulière, parce que toute la masse μ, égale à l'unité à la limite, vient se concentrer au point Q.

Le théorème d'unicité a été démontré précédemment, les points irréguliers formant un ensemble de capacité nulle (7).

SUBMARINE CABLE PROBLEMS SOLVED BY CONTOUR INTEGRATION.

By N. W. McLachlan.

1. *Introduction.*

The solution of the general case of a cable having constant resistance R, inductance L, capacitance C, and leakance G, all expressed per unit length, is usually obtained in the form of an expansion. Using his operational method, Heaviside was the first person to obtain a solution. F. W. Carter points out that Heaviside's result seems to be incorrect,* and proceeds to find a solution in the form of a double summation. By aid of complex integration, the integrand being expanded in Bessel functions, H. Jeffreys gives another form of series solution.† As this type of problem may arise in lectures on applied mathematics, it is proposed to show how a compact result can be got by contour integration, using a suitable transformation.

We take the case of a submarine cable, or an electrical transmission line, which is so long that reflection from the far end can be neglected at any intermediate point sufficiently remote from that end.‡ If E and I are the potential difference and current, respectively, at a point distant x from the origin or sending end, the equations to be solved are §

$$\frac{\partial E}{\partial x} = -\left(RI + L\frac{\partial I}{\partial t}\right), \quad \text{.....(1)}$$

$$\frac{\partial I}{\partial x} = -\left(GE + C\frac{\partial E}{\partial t}\right). \quad \text{.....(2)}$$

At $t=0$, a battery of voltage E_0 is connected to the cable at the sending end $x=0$.

2. *Current at any point x.*

Writing $p \equiv \partial/\partial t$ in (1), (2) and solving for I, the operational expression for the current is ‖

$$I = \frac{A(p+2b)}{\sqrt{(p+2a)(p+2b)}}\, e^{-\frac{x}{v}\sqrt{(p+2a)(p+2b)}}, \quad \text{.....(3)}$$

$$= \frac{A(p+2b)}{\sqrt{(p+\alpha)^2-\beta^2}}\, e^{-\frac{x}{v}\sqrt{(p+\alpha)^2-\beta^2}}, \quad \text{.....(4)}$$

* *Proc. Roy. Soc. A.* 156, 1, 1936.

† *Operational Methods in Mathematical Physics*, 2nd edition (1931).

‡ This avoids the necessity for assuming that the cable is of infinite length, in which case the steady state can never be attained!

§ McLachlan, *Bessel Functions for Engineers*, pp. 107, 108.

‖ The second solution is omitted, since by hypothesis there is no reflected wave.

where $A=E_0\sqrt{C/L}$, $a=R/2L$, $b=G/2C$, $v=1/\sqrt{LC}$ the velocity of propagation along the cable, $\alpha=(a+b)$, $\beta=(a-b)$. We have now to interpret (4) in terms of t. By aid of the Bromwich integral,* we get

$$I=\frac{A}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{zt-\frac{x}{v}\sqrt{(z+\alpha)^2-\beta^2}}\frac{(z+2b)\,dz}{z\sqrt{(z+a)^2-\beta^2}}. \quad \ldots\ldots\ldots\ldots(5)$$

Writing $(z+\alpha)=\zeta$, and replacing ζ by z thereafter, the first integral in (5) becomes

$$I_1=\frac{Ae^{-at}}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{zt-\frac{x}{v}\sqrt{z^2-\beta^2}}\frac{dz}{\sqrt{z^2-\beta^2}}. \quad \ldots\ldots\ldots\ldots(6)$$

Substituting $z=u/(t-x/v)+\beta^2(t-x/v)/4u$ in (6) [see Appendix concerning this and (8) below], gives

$$I_1=\frac{Ae^{-at}}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{u+\beta^2(t^2-x^2/v^2)/4u}\frac{du}{u}, \quad \ldots\ldots\ldots\ldots(7)$$

$$=Ae^{-at}I_0(\beta\sqrt{t^2-x^2/v^2}). \quad \ldots\ldots\ldots\ldots(8)$$

The second integral in (5) is obviously

$$I_2=2b\int_{x/v}^{t} I_1\,dt. \quad \ldots\ldots\ldots\ldots(9)$$

Hence the current at a point distant x from the battery is by (5), (8), (9),

$$I=E_0\sqrt{C/L}$$

$$\left[e^{-at}I_0(\beta\sqrt{t^2-x^2/v^2})+2b\int_{x/v}^{t} e^{-at}I_0(\beta\sqrt{t^2-x^2/v^2})\,dt\right]. \ldots(10)$$

The first term in brackets indicates that a time x/v elapses before current reaches the point x, *i.e.* $v=1/\sqrt{LC}$ is the velocity of propagation. When $t=x/v$, the current at x rises precipitately to the value $E_0\sqrt{C/L}e^{-ax/v}$. At $x=0$, $t=0$, $I=E_0\sqrt{C/L}$, the cable there acting momentarily as a resistance of value $\sqrt{L/C}$. The current can be considered to travel along the cable with a vertical wave front, whose height decreases with increase in distance from $x=0$, according to the law $e^{-ax/v}$. The second term in (10) is $2b=G/C$ times the integral of the first term, G/C having dimension t^{-1}. This term expresses the gradual growth of current at $x(t>x/v)$. The integration indicates that the charging current to the cable between $x=0$ and x steadily decreases with increase in time, thereby permitting more current to flow past the point x. The current represented by (10) ultimately settles down asymptotically to a steady value. Thus when $t=\infty$ in (10), the first term vanishes $(\alpha>\beta)$, whilst by the Appendix the value of the second term is

$$\frac{e^{-\frac{x}{v}\sqrt{\alpha^2-\beta^2}}}{\sqrt{\alpha^2-\beta^2}}=\sqrt{(LC/GR)}\,e^{-x\sqrt{GR}}. \quad \ldots\ldots\ldots\ldots(11)$$

* *Proc. L.M.S.* (2), 15, 401, 1916.

Consequently when $t=\infty$, (10) yields

$$I=E_0\sqrt{G/R}\,e^{-x\sqrt{GR}}, \quad \text{.....(12)}$$

which represents the leakage current through the insulation. This can be calculated for an actual cable using data given in § 4.

The behaviour of the first term in (10) depends upon the value of x. Near the origin the current arising from this term decreases with increase in $t(t>x/v)$. Beyond a certain value of x, the current reaches a maximum value as t increases, but sinks asymptotically to zero thereafter as $t\to\infty$.

3. *Potential difference at any point x.*

From (1) and (2) the operational expression for the P.D. is

$$E=E_0e^{-\frac{x}{v}\sqrt{(p+\alpha)^2-\beta^2}}, \quad \text{.....(13)}$$

the corresponding contour integral being

$$E=\frac{E_0}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{zt-\frac{x}{v}\sqrt{(z+\alpha)^2-\beta^2}}\frac{dz}{z}. \quad \text{.....(14)}$$

Differentiating (14) with respect to t, gives

$$\frac{dE}{dt}=\frac{E_0}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{zt-\frac{x}{v}\sqrt{(z+\alpha)^2-\beta^2}}\,dz, \quad \text{.....(15)}$$

and

$$\int\left(\frac{dE}{dt}\right)dx=-\frac{E_0v}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{zt-\frac{x}{v}\sqrt{(z+\alpha)^2-\beta^2}}\frac{dz}{\sqrt{(z+\alpha)^2-\beta^2}}+\xi(t), \quad (16)$$

$$=-E_0ve^{-\alpha t}I_0(\beta\sqrt{t^2-x^2/v^2})+\xi(t), \quad \text{.....(17)}$$

by (5)–(8), $\xi(t)$ being a function of t. Differentiating (17) with respect to x, we have

$$\frac{dE}{dt}=E_0\frac{\beta x}{v}e^{-\alpha t}\frac{I_1(\beta\sqrt{t^2-x^2/v^2})}{\sqrt{t^2-x^2/v^2}}, \quad \text{.....(18)}$$

so

$$E=E_0\left[\frac{\beta x}{v}\int_{x/v}^{t}e^{-\alpha t}\frac{I_1(\beta\sqrt{t^2-x^2/v^2})}{\sqrt{t^2-x^2/v^2}}\,dt\right]+\chi(x), \quad \text{.....(19)}$$

where $\chi(x)$ is a function of x, determined as follows: when $t=\infty$, the P.D. at x is by (12)

$$E=E_0-R\int_0^x E_0\sqrt{G/R}\,e^{-x\sqrt{GR}}\,dx=E_0e^{-x\sqrt{GR}}. \quad \text{.....(20)}$$

Omitting E_0, the value of the first member on the R.H.S. of (19) is (by § 4 in the Appendix)

$$e^{-(x/v)\sqrt{\alpha^2-\beta^2}}-e^{-\alpha x/v}=e^{-x\sqrt{GR}}-e^{-\alpha x/v}. \quad \text{.....(21)}$$

Consequently $\chi(x) = E_0 e^{-\alpha x/v}$, so (19) can be written

$$E = E_0\left[e^{-\alpha x/v} + \frac{\beta x}{v}\int_{x/v}^{t} e^{-\alpha t}\frac{I_1(\beta\sqrt{t^2 - x^2/v^2})}{\sqrt{t^2 - x^2/v^2}}\,dt\right]. \quad \ldots\ldots(22)$$

When $x = 0$, $E = E_0$ the battery voltage, whilst for $t = \infty$ formula (20) is obtained, so the boundary conditions are satisfied. Formula (22) applies when $t \geqslant x/v$. When $t = x/v$, the P.D. suddenly jumps to the value $E_0 e^{-\alpha x/v}$. It increases with increase in time, by virtue of the integral member of (22), and approaches the value $E_0 e^{-x\sqrt{GR}}$ asymptotically.

4. *Practical data.*

The following data for a cable may be of interest :

$R = 1{\cdot}8$ ohm per nautical mile.
$L = 10^{-1}$ henry " " "
$C = 4 \times 10^{-7}$ farad " " "
$1/G = 10^7$ ohms " " "

$a = \dfrac{R}{2L} = 9$ sec.$^{-1}$, the time constant.

$\dfrac{1}{\sqrt{LC}} = 5000$ nautical miles per second, the velocity of propagation.

$\sqrt{L/C} = 500$ ohms, the effective impedance at the sending end at the instant when the battery is applied.

Sending voltage $= 40$.

Initial current at sending end $= 40/500 = 8 \times 10^{-2}$ ampere,

" " at $x = 3500$ N.M. $= 146$ microamperes, $0{\cdot}7$ second after the sending voltage is applied.

The wave form of the voltage applied to the cable is that of Heaviside's unit function. It can be resolved into an infinite frequency spectrum. For the initial cable currents to have the above values (vertical wave front), the cable coefficients R, L, C, G must be independent of frequency. This is not so in practice, since R increases whilst L decreases with rise in frequency. As a consequence the wave front is not vertical, and the current at $x = 3500$ N.M. rises from zero at $t = 0{\cdot}7$ sec. approx., with a finite slope.

Appendix.

1. The transformation

$$z = u/(t - x/v) + \beta^2(t - x/v)/4u, \quad \ldots\ldots(1)$$

or

$$u = \tfrac{1}{2}(t - x/v)(z + \sqrt{z^2 - \beta^2}), \quad \ldots\ldots(2)$$

with t, x, v real and positive, $t > x/v$.

If the contour in the z plane crosses the real axis at $c_1 > \beta$ on the right of the origin, $u = \tfrac{1}{2}(t - x/v)(c_1 + \sqrt{c_1^2 - \beta^2})$ which is positive.

For large z, (2) gives $u \sim (t-x/v)z$. Thus if the contour in the z plane is $c \pm i\infty$ with c fixed or variable, the contour in the u plane can be drawn in conformity therewith and the two contours are equivalent.

2. The contour integral for the modified Bessel function is

$$I_0(\beta\sqrt{t^2-x^2/v^2}) = \frac{1}{2\pi i}\int_{-\infty}^{(0+)} e^{u+\beta^2(t^2-x^2/v^2)/4u}\frac{du}{u}, \quad \ldots\ldots\ldots(3)$$

the contour being a loop round the origin with the negative real axis as barrier. If the contour $c \pm i\infty$ is jointed to this by infinite arcs in the second and third quadrants, the complete contour contains no singularity. It is easy to show that the integrals along the arcs vanish, so that the contour in (3) can be replaced by $c \pm i\infty$.

3. To evaluate the integral in (10) when $t=\infty$, we note from (6), (8) that

$$I_0(\beta\sqrt{t^2-x^2/v^2}) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{zt-\frac{x}{v}\sqrt{z^2-\beta^2}}\frac{dz}{\sqrt{z^2-\beta^2}}. \quad \ldots\ldots\ldots(4)$$

Now by the Mellin inversion theorem,*

if $$\phi(z) = \int_{x/v}^{\infty} e^{-zt} f(t^2-x^2/v^2)\,dt, \quad \ldots\ldots\ldots(5)$$

then $$f(t^2-x^2/v^2) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{zt}\,\phi(z)\,dz. \quad \ldots\ldots\ldots(6)$$

Hence by (4), (5), (6)

$$\int_{x/v}^{\infty} e^{-\alpha t} I_0(\beta\sqrt{t^2-x^2/v^2})\,dt = \frac{e^{-\frac{x}{v}\sqrt{\alpha^2-\beta^2}}}{\sqrt{\alpha^2-\beta^2}}. \quad \ldots\ldots\ldots(7)$$

4. To evaluate the integral in (19) when $t=\infty$, differentiate both sides of (7) with respect to x, thereby obtaining

$$-\frac{e^{-\alpha x/v}}{v} - \frac{\beta x}{v^2}\int_{x/v}^{\infty} e^{-\alpha t}\frac{I_1(\beta\sqrt{t^2-x^2/v^2})\,dt}{\sqrt{t^2-x^2/v^2}} = -\frac{e^{-\frac{x}{v}\sqrt{\alpha^2-\beta^2}}}{v}, \quad \ldots\ldots(8)$$

The integral in (8) is convergent. N. W. McL.

1176. In the year that Piazzi discovered Ceres seven planets were known. ... In the same year the Prussian Philosopher Hegel wrote upbraiding scientists for the neglect of Philosophy. He illustrated his disapproval by the amount of time astronomers wasted in looking for a new planet when philosophy clearly established the only possible number as seven. Since then other planets have been discovered.—L. Hogben, *Mathematics for the Million*, p. 221. [Per Mr. T. S. Venkatraman.]

* See Courant and Hilbert, *Methoden der Mathematischen Physik*, pp. 87, 88 (1931), or Titchmarsh, *Theory of Fourier Integrals*, § 1.5, (1937). The form used here is a particular case where all the singularities are on the left of $c \pm i\infty$. See also Bromwich, *loc. cit.*

ON THE PARTITION OF SETS OF POINTS ON A RECTANGULAR HYPERBOLA.

By A. A. Krishnaswami Ayyangar.

Introductory

We propose to study in this paper the properties of a set of points on a proper rectangular hyperbola. We call this set an H-set, which is characterised as pure or mixed according as the points of the set belong to one branch or to both the branches of the hyperbola.

Four points are said to form a *convex* set when no one of them lies within the triangle whose vertices are the other three. A *convex set* is one of which every subset of four points is a convex set; *e.g.* a set of points on a circle. Every subset of a convex set is also convex. The points of a convex set admit of being joined suitably so as to form a closed polygon. This polygon is called a *convex polygon*. The *angle* at any point *of a convex set* is the interior angle at that point of the derived convex polygon. The linear segment between two points of a set is a chord of the set and when it is of maximum length, its extremities are called the *extreme points* of the set. A set may have more than one pair of extreme points. A set with a unique pair of extreme points is *singular*. A set consisting of only one point is a *unit set*.

We assume throughout this paper that no three points of a set are collinear.

Fundamental Partitions

1. Any straight line parallel to an asymptote of a rectangular hyperbola (briefly r.h.) cuts the curve only at one point not at infinity. Two such lines one parallel to each asymptote give rise to no empty quadrant or a single empty one according as their intersection lies within or outside the region enclosed by the curve and its asymptotes. If the parallels be drawn through a point on the curve itself, there is again only one empty quadrant such that its vertically opposite quadrant contains that branch of the curve to which the given point does not belong. (The empty quadrants are marked E_1, E_2, E_3 corresponding to the points R_1, R_2, R_3; for R_4 there is no empty quadrant. *Vide* Fig. 1.)

1.1. *A chord of one branch of a r.h.* divides that branch into three segments and *subtends an acute angle at every point on the undivided branch*, since the former lies entirely within the quadrant opposite to the empty quadrant formed by the parallels to the asymptotes through this latter point. For a similar reason, *the same chord subtends an obtuse angle at every point of the finite segment and an acute angle at every point of the two infinite segments of the divided branch*. Further the finite segment lies on the same side of the chord as the undivided branch while the two infinite segments lie on the other side.

1.2. A circle having as its diameter a chord of one branch of a r.h. partitions the points of the curve into *four* sets such that (1) the finite segment of the divided branch lies entirely within the circle, (2) the undivided branch lies entirely outside the circle and on the

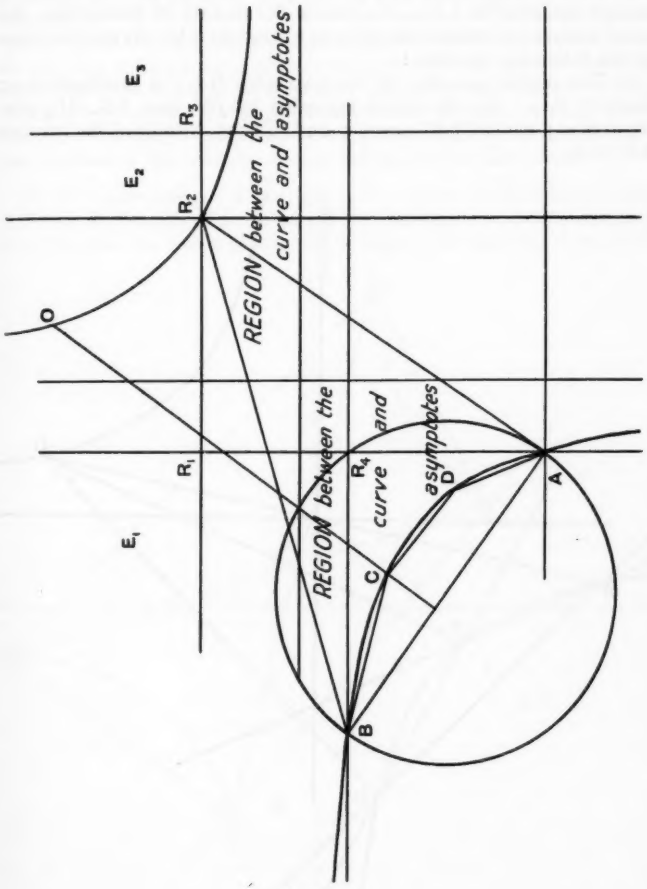


FIG. 1.

same side of the chord as the finite segment, while on the other side lie the remaining segments of the divided branch also outside the circle and (3) no point of the curve except the ends of the chord lies on the circle. (*Vide* Fig. 1.)

1.21. *Every triangle inscribed in the same branch of a r.h. is obtuse-angled. An acute-angled triangle inscribed in a r.h. has two vertices on one branch and one vertex on the other. The hypotenuse of a right-angled triangle inscribed in a r.h. has its extremities on different branches.*

2. Before proceeding to the partitions made by the side-lines of a triangle inscribed in a r.h., we denote, for facility of description, the seven regions into which the sides of a triangle ABC divide the plane by the following symbols :

(i) The region bounded by the two sides BA, CA produced is α; similarly β, γ ; (ii) the region bounded by BC, and AB, AC, produced is α' ; similarly β', and γ' ; and (iii) the interior of the triangle ABC is δ.

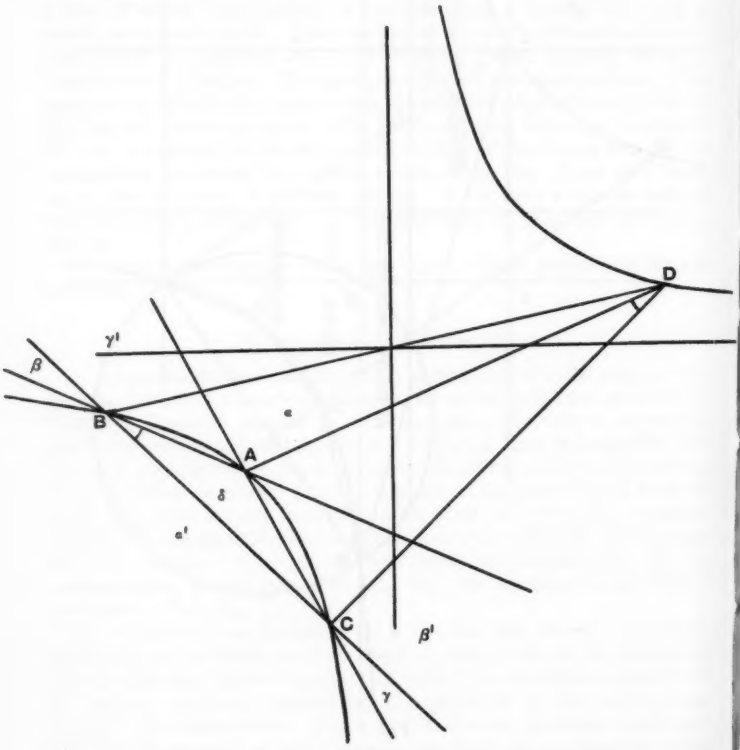


FIG. 2.

2.1. The side-lines of a triangle ABC (where $\widehat{BAC}$ is obtuse) inscribed in a branch of a r.h., partition the points of the curve into *five* sets such that (1) there is one undivided branch in α, (2) the

divided branch of four segments has its infinite segments in α', one finite segment in β' and another in γ', and (3) the empty regions are β, γ, and δ.

2.11. *Any four points on the same branch form a convex set.* The orthocentre of a triangle inscribed in any branch lies on the other branch.

2.12. *Any three points on one branch and a fourth point on the other do not form a convex set; of the three points on the same branch, that one at which the obtuse angle is situated lies within the triangle formed by the other two and the fourth point on the other branch.*

2.13. If a triangle inscribed in a r.h. contains a point of the curve the vertices of the triangle do not belong to the same branch.

2.2 The side-lines of a triangle ABC (where $\widehat{BAC}$ is acute) with B,C, on one branch of a r.h. and A on the other, partition the curve into five sets, such that (1) the finite segment of the branch to which

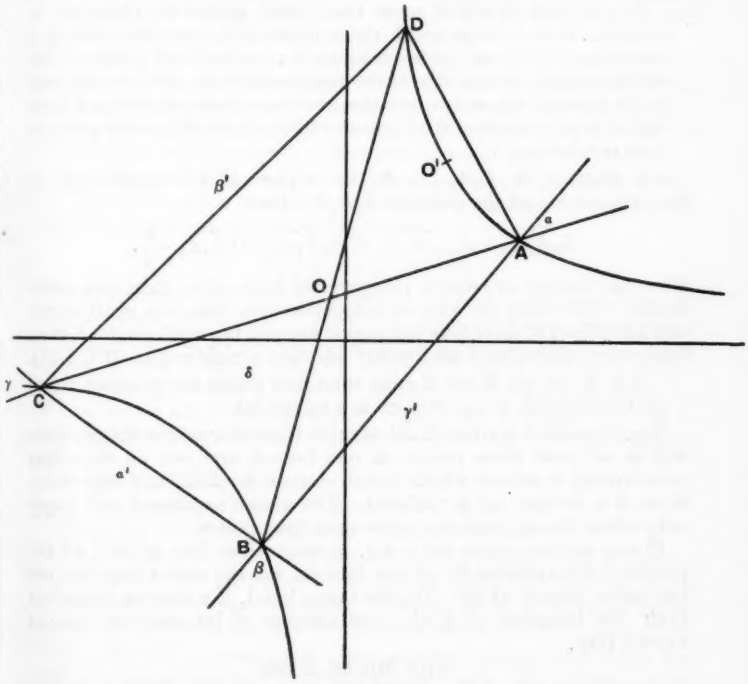


FIG. 3.

B, C, belong lies in δ and the two infinite segments of the same branch lie respectively in β and γ, (2) the two infinite segments of the other

branch lie severally in β', γ' and (3) the empty regions are α and α'.

2.21. *Any two points of one branch and any two points of the other form a convex set such that the chord of either branch subtends an acute angle at the centre of the r.h.*

The Pure H-set

3. If any number of points be taken on the same branch of a r.h., they form a convex polygon, which is inscribed in a hyperbolic segment. Since the circle on the chord of this segment as diameter encloses the entire segment as well as the polygon by 1.2, this chord is the maximum chord of the set of points. The set has evidently no other maximum chord and is therefore singular.

I have proved elsewhere * that if the circle having as diametrically opposite points two vertices of a convex polygon, encloses the entire polygon, it is the circle of least area enclosing it.

3.1. A pure H-set of more than three points is, therefore, a singular convex set in which the extreme points are the ends of a diameter of the least circle enclosing it; and all the points of the set lie entirely on one side of the maximal chord, and are enclosed in the triangle whose two vertices are the extreme points and third vertex is any point on that branch of the r.h. to which the pure set does not belong.

3.2. Further, if $A_1A_2 \ldots A_n$ be a pure H-set segment and O the orthocentre of the triangle $A_2A_1A_n$, then

$$A_2\widehat{A_1}A_n + A_{n-1}\widehat{A_n}A_1 < A_2\widehat{A_1}A_n + OA_n\widehat{A_1} = \frac{\pi}{2}.$$

Since no triangle or convex polygon can have more than two acute angles, when there are two such together less than one right angle, *any pure H-set of more than two points has just two acute angles; these angles are adjacent and are together less than a right angle.* (Cf. 1.21).

3.3. A convex H-set of more than four points is a pure set; and an H-set which is not convex is a mixed set.

For, if a mixed convex H-set of more than four points exists, there will be at least three points on one branch and one on the other constituting a subset which is not convex by 2.12, and our definition of a convex set is violated. The above argument will apply only when the set contains more than four points.

If any convex curve cut a r.h. in more than four points, all the points of intersection lie on one branch, *i.e.* the curve does not cut the other branch at all. On the other hand, if a convex curve cut both the branches of a r.h., the number of intersections cannot exceed four.

The Mixed H-set

4. Every mixed H-set is made up of two pure H-sets, each of which has two extreme points, unless either is a unit set.

* *Journal of the Indian Mathematical Society*, 1929, Vol. XVIII, No. 5, pp. 101-6.

Let (A, B), (C, D) be the extreme points, respectively, of two pure sets, H_1, H_2 which are not unit sets and constitute the given mixed set. By 3.1 they are bounded, H_1 by the triangles CAB, DAB, and H_2 by the triangles ACD, BCD. If the cross-joins of the extreme pairs meet in O, H_1 will be bounded by the triangle OAB, and H_2 by the triangle OCD, and thus the mixed set is easily partitioned off into pure sets belonging to different branches.

By 2.21, (A, B, C, D) is a convex set; the convex quadrilateral derived from it encloses the entire H-set, so that there is no other convex subset of the H-set with the same property. We call this the *bounding* subset.

4.1. Every mixed H-set of more than four points has a unique bounding subset of three or four points; in the former case, all points of the H-set except one (without which the rest will be convex) lie on one branch and in the latter case, the two component pure sets are enclosed in either pair of vertically opposite angles formed by the cross-joins of the four points, the specific pair being determined by the position of the fifth point of the set.

A pure H-set of more than four points has no bounding subset. A pure H-set and a mixed one of more than four points, therefore, are distinguished from each other in two ways: (1) The pure set is convex and the mixed one is not; (2) the pure set has no bounding convex sub-set of four points, which a mixed set has.

4.2. When an H-set of four points is not an orthocentric tetrad, we can adjoin to it a fifth point which is the orthocentre of one of the triangles of the set, so that 4.1 applies.

The Four-point Set

5. A four-point set differs in certain essential respects from a set of more than four points. Every four-point set is an H-set unless it belongs to a pair of perpendicular lines without forming an orthocentric tetrad. This is not true of a set of more than four points. The results of 3.3 and 4.1 have to be modified for the four-point set. We assume a set of four points through which a proper r.h. can be drawn. The partitions of this set between the two branches are either:

(1) (4 : 0) which gives a convex set by 2.11;
(2) (3 : 1) not leading to a convex set by 2.12; or
(3) (2 : 2) which again gives a convex set by 2.21.

5.1. A convex set of four points is either a pure H-set or a mixed H-set with two points on each branch, while a set of four points which is not convex is a mixed H-set made up of one pure subset of three points on one branch and a unit subset on the other branch.

5.2. By 3.2, a pure convex set of four points must have two adjacent angles together less than a right angle. We will now prove that *a convex H-set of four points having two adjacent angles together less than a right angle, is pure.*

Let $ABCD$ be a convex quadrilateral in which $\widehat{DAB} + \widehat{CBA} < 90°$.

If O is the orthocentre of the triangle ABC, O lies on the same side of AB as C and since $O\hat{A}B = 90° - A\hat{B}C > D\hat{A}B$, and

$$O\hat{B}A = 90° - C\hat{A}B > 90° - D\hat{A}B > A\hat{B}C,$$

C and D lie within the triangle OAB. Thus (A, B, C, D, O) is a mixed H-set with (O, A, B) as the bounding subset. By 4.1, O lies on one branch and A, B, C, D on the other. (*Vide* Fig. 1.)

5.3. In any convex quadrilateral in which no pair of opposite sides cut at right angles, either there is a unique pair of adjacent angles together less than a right angle or any two adjacent angles are together greater than a right angle. We have proved that the vertices, in the former case, lie on the same branch of a rectangular hyperbola. Hence in the latter case, two vertices must lie on one branch and two on the other. For a further specification of these vertices, we proceed as follows:

Let $ABCD$ be the given convex quadrilateral (not a parallelogram *), in which AC, BD cut at O; let BOC be acute. If any one of the four angles ADB, ACB, CAD, CBD be obtuse or right, neither (C, D) nor (A, B) lie on the same branch by 1.1. We shall next consider the only remaining alternative, viz. when all these four angles are acute. Since all the angles of the quadrilateral cannot be obtuse, let $C\hat{D}A$ be acute. Then the triangle ACD is acute-angled and its orthocentre, say O', lies within the triangle AOD. Applying 4.1, we see that (A, B, C, D) is the bounding sub-set of the mixed set (A, B, C, D, O') which, therefore, consists of two pure subsets (A, O', D) and (B, C) belonging to different branches. Therefore (B, C), (A, D) belong to different branches in every case, and each of the sides BC, AD subtends an acute angle at O. The cases where any other angle of the quadrilateral is acute lead to the same result. (*Vide* Fig. 3.)

The necessary and sufficient condition that two adjacent vertices of the quadrilateral in the second case should lie on opposite branches (or the same branch) is that the chord joining them should subtend an obtuse (or acute) angle at the intersection of the diagonals.

5.31. If the tangents at A and B, two points on a r.h., meet in C, then $A\hat{C}B$ is obtuse or acute according as A, B lie on the same branch or on different branches.

5.4. For the decomposition of an orthocentric tetrad into pure H-sets on different branches, we have the following theorem.

The r.h. circumscribing an acute-angled triangle ABC has (B, C) on one branch and A on the other, if its centre lies on the arc of the nine-point circle which subtends that angle of the pedal triangle whose vertex lies on the side BC; and conversely.

* When $ABCD$ is a parallelogram, O becomes the centre of the circumscribed r.h., with B, C on one branch, and A, D on the other, provided $B\hat{O}C$ be acute (*Vide* 2.21).

If D is the orthocentre of the triangle ABC and α the centre of the r.h., the point D' of the hyperbola diametrically opposite to D lies on the circumcircle of ABC. This is a well-known result. If α lies as specified in the hypothesis, DD' lies within the angle vertically opposite to BDC, and D lies within the triangle $BD'C$. Since the subset (A, D', B, C) is obviously the bounding subset of (A, B, C, D, D') we have, by 4.1, (B, C, D) on one branch and A on the other.

Similarly, if α lies on the arc of the nine-point circle subtending the angle of the pedal triangle, whose vertex lies on CA (or AB), the r.h. will have A, C, D (or A, B, D) on one branch and B (or C) on the other. When α coincides with a vertex of the pedal triangle, the r.h. degenerates into two perpendicular lines. By indirect reasoning, the converses are also true.

5.5. Another interesting four-point set is (A, B, C, D) in which $\widehat{ABC} = \widehat{ADC}$. If the set be convex, 5.3 applies. If the set be not convex, we can prove that the only possible points on the same branch of the r.h. through the set are (A, B, C) or (A, C, D), according as DC is greater or less than CB.

It is easily seen that the centre of the r.h. is the middle point of BD, which lies on the nine-point circles of all the triangles of the set. Hence B, D lie on different branches. Again, since BD cuts the r.h., the perpendicular bisector of BD does not cut it, so that all points on one branch lie on one side of this line and those on the other branch on the other side; while (A, B, C) or (A, C, D) lie on the same side of the perpendicular bisector of BD, according as DC is greater or less than CB. (*Vide* Fig. 2.)

If $BC<CD$, $\widehat{BAC}$ is obtuse if A lies within the triangle BCD, by 2.12, and therefore $BC>BA$. But there is nothing to prevent $AD=CD$, and in this case, $\widehat{BCD}+\widehat{BAD}>\pi$. These results may be restated as an elementary theorem in " congruence of triangles " :

Two triangles ABC, DEF are congruent, if $BC=EF$, $\hat{B}-\hat{C}=\hat{E}-\hat{F}$, and $AB=DE$, provided $\hat{A}+\hat{D}<\pi$ when $AB>AC$.*

5.6. Finally, we consider the decomposition of the *general* (*i.e.* not orthocentric) non-convex tetrad (A, B, C, D) into pure H-sets on different branches. (*Vide* 4.2.)

Let D lie within the triangle ABC and let AD, BD, CD meet BC, CA, AB respectively in X, Y, Z. We remark that $\widehat{BXD}, \widehat{CYD}, \widehat{AZD}$ *cannot all be obtuse, nor all acute*, so that of the three pairs $(\widehat{BYC}, \widehat{BZC})$, $(\widehat{CZA}, \widehat{CXA})$, $(\widehat{AXB}, \widehat{AYB})$ only one can consist of an obtuse and an acute angle. *If*, for example $(\widehat{BYC}, \widehat{BZC})$ *be such a pair, i.e. one obtuse and the other acute, then* (B, C, D) *lie on one, and* A *on the other branch of the r.h. through* A, B, C, D. A. A. K. A.

* *Vide* the author's *New Geometry*, p. 207 (Srinivasa Varadachari & Co., Madras, India, 1928).

MORLEY'S TRIANGLE.

By W. J. Dobbs.

I. There are three ways of trisecting an angle bounded by OA and OB ($=3\alpha$ *or* $3\alpha+2\pi$ *or* $3\alpha+4\pi$), namely:

(i) OX_1 and OY_1,
(ii) OX_2 and OY_2,
(iii) OX_3 and OY_3,

as shown in Fig. 1.

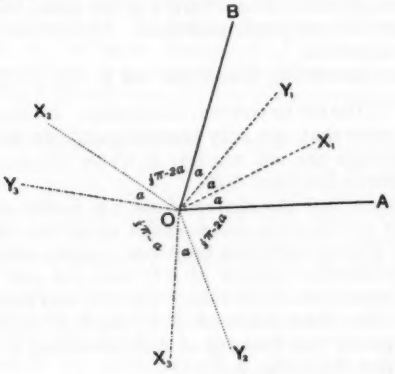


Fig. 1.

It is well known that if the Euclidean angles of a triangle ABC are trisected as in Fig. 2, then the triangle xyz (commonly quoted as Morley's Triangle) is equilateral. But angles CAB, ABC, BCA may be trisected each in three ways, so that there are really 27 "Morley" triangles to be considered. The purpose of the present paper is to show that 18 of these 27 triangles are equilateral.

II. Let $A=3\alpha$, $B=3\beta$, $C=3\gamma$, so that $\alpha+\beta+\gamma=\frac{1}{3}\pi$, and let d denote the circum-diameter of the triangle ABC.

From the identity $\sin 3\theta = 4\sin\theta\sin(\frac{1}{3}\pi+\theta)\sin(\frac{1}{3}\pi-\theta)$ we have

$$Cx = BC\sin\beta/\sin(\beta+\gamma) = 4d\sin\alpha\sin\beta\sin(\tfrac{1}{3}\pi+\alpha).$$

Similarly, interchanging α and β,

$$Cy = 4d\sin\beta\sin\alpha\sin(\tfrac{1}{3}\pi+\beta);$$

hence
$$Cx/Cy = \sin(\tfrac{1}{3}\pi+\alpha)/\sin(\tfrac{1}{3}\pi+\beta).$$

Also, as the angle xCy is γ, and $(\frac{1}{3}\pi+\alpha)+(\frac{1}{3}\pi+\beta)+\gamma=\pi$, it follows that the angles of the triangle Cxy are respectively γ, $\frac{1}{3}\pi+\beta$, $\frac{1}{3}\pi+\alpha$; so that xy is inclined to BC at angle $(\frac{1}{3}\pi+\beta)-\gamma=\alpha+2\beta$.

Similarly, interchanging β and γ,

xz is inclined to CB at angle $\alpha+2\gamma$;

hence the angle $zxy = \pi - (\alpha + 2\beta) - (\alpha + 2\gamma) = \frac{1}{3}\pi$;

Similarly, the other angles of the triangle xyz are each $\frac{1}{3}\pi$.

Therefore xyz is an equilateral triangle.

[Notice that xy is inclined to Bx at angle $\alpha + \beta = \frac{1}{3}\pi - \gamma$.]

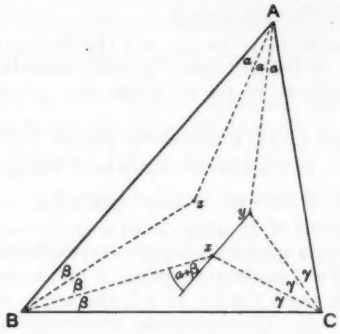


FIG. 2.

III. Now, instead of drawing Bx so that the angle $CBx = \frac{1}{3}B$, let us draw it so that the angle $CBx = \frac{1}{3}(B + 2\pi)$ and make a *corresponding* change in Cx. We thus obtain a point ξ on the side of BC remote from A such that the angle $CB\xi = \frac{1}{3}\pi - \beta$ and the angle $BC\xi = \frac{1}{3}\pi - \gamma$. *Corresponding* constructions determine the points η and ζ. (Fig. 3.)

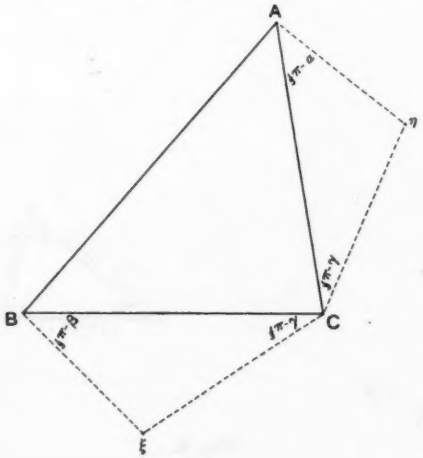


FIG. 3.

In this case

$$C\xi = BC \sin(\tfrac{1}{3}\pi - \beta)/\sin(\tfrac{1}{3}\pi + \alpha) = 4d \sin\alpha \sin(\tfrac{1}{3}\pi - \alpha) \sin(\tfrac{1}{3}\pi - \beta).$$

Similarly, interchanging α and β,

$$C\eta = 4d \sin\beta \sin(\tfrac{1}{3}\pi - \beta) \sin(\tfrac{1}{3}\pi - \alpha);$$

hence $$C\xi/C\eta = \sin\alpha/\sin\beta.$$

Also, as the angle $\xi C\eta = \frac{2}{3}\pi + \gamma$, and $(\frac{2}{3}\pi + \gamma) + \alpha + \beta = \pi$, it follows that the angles of the triangle $C\xi\eta$ are respectively $\frac{2}{3}\pi + \gamma, \beta, \alpha$; so that $\xi\eta$ is inclined to BC at angle $(\frac{1}{3}\pi - \gamma) + \beta = \alpha + 2\beta$, and is parallel to xy.

Similarly $\eta\zeta$ and $\zeta\xi$ are respectively parallel to yz and zx;

hence $\xi\eta\zeta$ is also an equilateral triangle.

[Notice that $\xi\eta$ is inclined to ξC at angle β.]

IV. Again, instead of drawing Bx so that the angle $CBx = \frac{1}{3}B$, let us draw it so that the angle $CBx = \frac{1}{3}(B + 4\pi)$ and make a *corresponding* change in Cx. We thus obtain a point X on the same side of BC as A, such that the angle $CBX = \frac{1}{3}\pi + \beta$, the angle $BCX = \frac{1}{3}\pi + \gamma$ and the angle $BXC = \alpha$. *Corresponding* constructions determine the points Y and Z. (Fig. 4.)

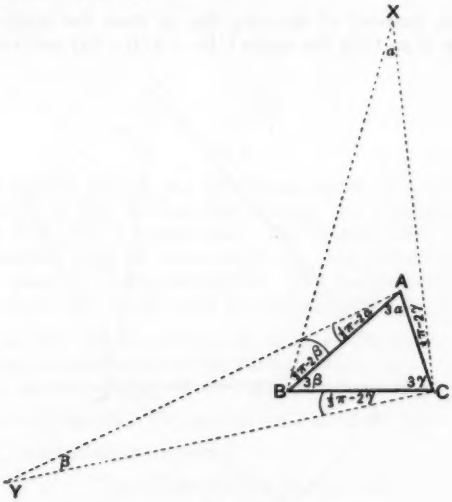


FIG. 4.

In this case,

$$CX = BC \sin(\tfrac{1}{3}\pi + \beta)/\sin\alpha = 4d \sin(\tfrac{1}{3}\pi + \alpha) \sin(\tfrac{1}{3}\pi - \alpha) \sin(\tfrac{1}{3}\pi + \beta).$$

Similarly, interchanging α and β,

$$CY = 4d \sin(\tfrac{1}{3}\pi + \beta) \sin(\tfrac{1}{3}\pi - \beta) \sin(\tfrac{1}{3}\pi + \alpha);$$

hence $$CX/CY = \sin(\tfrac{1}{3}\pi - \alpha)/\sin(\tfrac{1}{3}\pi - \beta).$$

Also, as the angle $XCY = \frac{2}{3}\pi - \gamma$, and $(\frac{2}{3}\pi - \gamma) + (\frac{1}{3}\pi - \alpha) + (\frac{1}{3}\pi - \beta) = \pi$, it follows that the angles of the triangle CXY are respectively $\frac{2}{3}\pi - \gamma$, $\frac{1}{3}\pi - \beta$, $\frac{1}{3}\pi - \alpha$; so that YX is inclined to BC at angle $(\frac{1}{3}\pi - \alpha) + (\frac{1}{3}\pi - 2\gamma) = \alpha + 2\beta$ and is parallel to xy.

Similarly ZY and XZ are respectively parallel to yz and zx; thus

XYZ is also an equilateral triangle.

[Notice that XY is inclined to XB at angle γ.]

V. We have now three "Morley" triangles, xyz, $\xi\eta\zeta$, XYZ, all equilateral, having their sides respectively parallel.

The trisectors of angle CAB are Ay and Az *or* ηA and ζA *or* YA and ZA.

The trisectors of angle ABC are Bz and Bx *or* ζB and ξB *or* ZB and XB.

The trisectors of angle BCA are Cx and Cy *or* ξC and ηC *or* XC and YC.

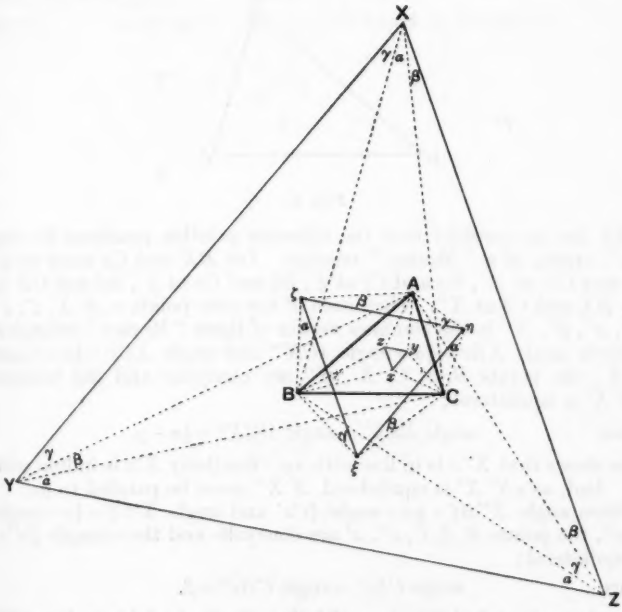


FIG. 5.

Returning to the original "Morley" triangle xyz, we may keep Bz and Bx as the trisectors of B and at the same time Cx and Cy as the trisectors of C, while we vary the trisectors of angle CAB.

In this way we obtain three " Morley " triangles having a common vertex at x, namely :

(i) the triangle xyz ;

(ii) the triangle having its " x " vertex at x, its " y " vertex at the intersection of Cy and $A\eta$, and its " z " vertex at the intersection of $A\zeta$ and Bz ;

(iii) the triangle having its " x " vertex at x, its " y " vertex at the intersection of Cy and AY, and its " z " vertex at the intersection of AZ and Bz.

It is obvious that (ii) and (iii) cannot both be equilateral.

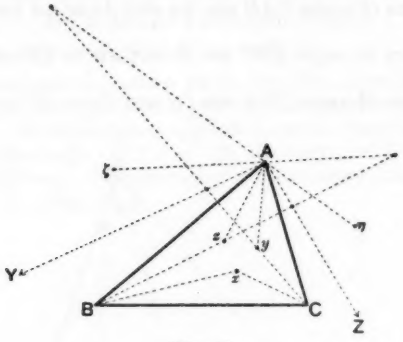


FIG. 6.

VI. Let us consider now the different possible positions for the " x " vertex of a " Morley " triangle. Let BX and Cx meet at x', Bx and CX at x'', Bx and $C\xi$ at ξ', $B\xi$ and Cx at ξ'', $B\xi$ and CX at X', BX and $C\xi$ at X''. Then each of the nine points x, ξ, X, x', ξ', X', x'', ξ'', X'' is the common vertex of three " Morley " triangles.

Since angle $XBx = \frac{1}{3}\pi =$ angle xCX'' and angle $XCx = \frac{1}{3}\pi =$ angle xBX', the points B, x, C, X', X'' are concyclic and the triangle $xX''X'$ is equilateral ;

hence $$\text{angle } BxX'' = \text{angle } BCX'' = \tfrac{1}{3}\pi - \gamma.$$

This shows that $X''x$ is in line with xy. Similarly $X'x$ is in line with xz. And, as $xX''X'$ is equilateral, $X'X''$ must be parallel to yz.

Since angle $X''B\xi = \frac{1}{3}\pi =$ angle $\xi Cx'$ and angle $X'C\xi = \frac{1}{3}\pi =$ angle $\xi Bx''$, the points B, ξ, C, x'', x' are concyclic and the triangle $\xi x''x'$ is equilateral ;

hence $$\text{angle } C\xi x'' = \text{angle } CBx'' = \beta.$$

This shows that $\xi x''$ is in line with $\xi\eta$. Similarly $\xi x'$ is in line with $\xi\zeta$. And, as $\xi x''x'$ is equilateral, $x'x''$ must be parallel to $\zeta\eta$ and $X''X'$.

If $X''x$ meets $x'x''$ at H, the points x, C, x'', H are concyclic, and therefore angle $xCH =$ angle $xx''x'$ (*i.e.* angle $Bx''x'$) $=$ angle

$BCx' = \gamma$. This shows that H coincides with y. Similarly $X'x$ meets $x'x''$ at z. Thus $x''x'$ is in line with yz.

Since angle $\xi''BX = \frac{1}{3}\pi =$ angle $\xi''CX$ and angle $\xi'CX = \frac{1}{3}\pi =$ angle $\xi'BX$, the points B, C, ξ', X, ξ'' are concyclic and the triangle $X\xi'\xi''$ is equilateral;

hence angle $BX\xi'' =$ angle $BC\xi'' = \gamma$.

This shows that $X\xi''$ is in line with XY. Similarly $X\xi'$ is in line with YZ. And, as $X\xi''\xi'$ is equilateral, $\xi''\xi'$ must also be parallel to zy and $X''X'$.

If $\xi x''$ meets $\xi''\xi'$ at K, the points C, ξ', K, x'' are concyclic, and therefore angle $x''CK$ = angle $x''\xi'K$ = angle $Bx''x'$ = angle $BCx' = \gamma$;

hence angle $BCK = \frac{1}{3}\pi + 2\gamma$ and angle $ACK = \frac{1}{3}\pi - \gamma$.

This shows that K coincides with η. Similarly $\xi x'$ meets $\xi''\xi'$ at ζ. Thus $\xi'\xi''$ is in line with $\eta\zeta$.

If $X\xi''$ meets $X'X''$ at L, the points C, X, L, X'' are concyclic, and therefore angle $X''CL =$ angle $X''XL = \gamma$;

hence angle $BCL = \frac{1}{3}\pi - 2\gamma$ and angle $ACL = \frac{1}{3}\pi + \gamma$.

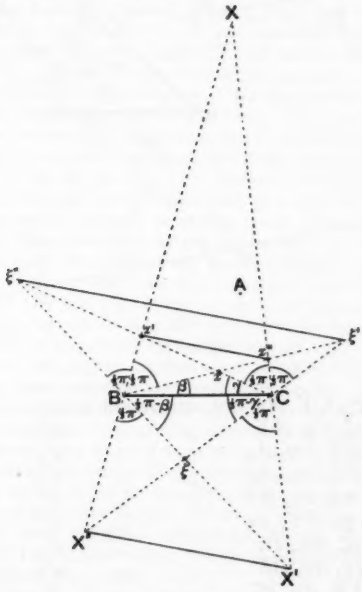


FIG. 7.

This shows that L coincides with Y. Similarly $X\xi'$ meets $X''X'$ at Z. Thus $X''X'$ is in line with YZ.

VII. We have proved that

$yz, \zeta\xi, BX, Cx$	are concurrent at		x',
$yz, \xi\eta, Bx, CX$	,,	,,	x'',
$\eta\zeta, ZX, Bx, C\xi$	,,	,,	ξ',
$\eta\zeta, XY, B\xi, Cx$	,,	,,	ξ'',
$YZ, zx, B\xi, CX$	,,	,,	X',
$YZ, xy, BX, C\xi$	,,	,,	X''.

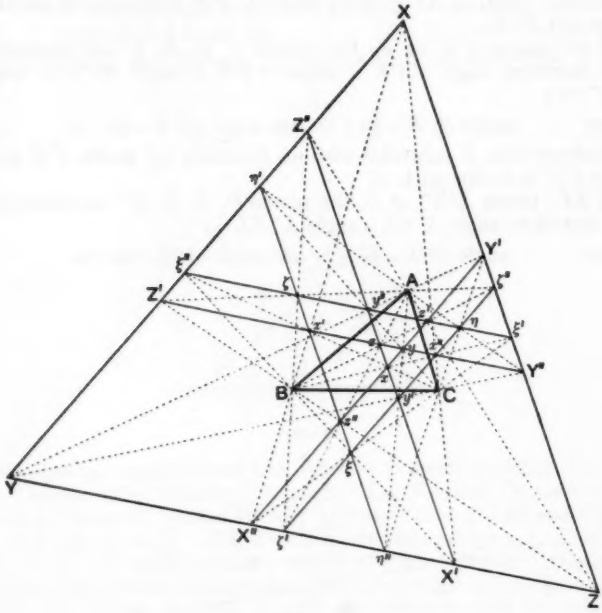


FIG. 8.

Similarly,

$zx, \xi\eta, CY, Ay$	are concurrent at		y',
$zx, \eta\zeta, Cy, AY$	,,	,,	y'',
$\zeta\xi, XY, Cy, A\eta$	,,	,,	η',
$\zeta\xi, YZ, C\eta, Ay$	,,	,,	η'',
$ZX, xy, C\eta, AY$	,,	,,	Y',
$ZX, yz, CY, A\eta$	,,	,,	Y'',
$xy, \eta\zeta, AZ, Bz$	,,	,,	z',
$xy, \zeta\xi, Az, BZ$	,,	,,	z'',
$\xi\eta, YZ, Az, B\zeta$	,,	,,	ζ',
$\xi\eta, ZX, A\zeta, Bz$	,,	,,	ζ'',
$XY, yz, A\zeta, BZ$	,,	,,	Z',
$XY, zx, AZ, B\zeta$	,,	,,	Z''.

The notation is such that any letter with one or two dashes can be easily located. For instance η' is situated in the line $\zeta\xi$, Z'' in the line XY, y'' in the line zx, ξ' in the line $\eta\zeta$.

VIII. It is now obvious that the following 18 " Morley " triangles are all equilateral :

$$xyz,\ xy''z',\ \xi\eta\zeta,\ \xi\eta''\zeta',\ XYZ,\ XY''Z',$$
$$x'yz'',\ x'\eta'Z',\ \xi'\eta\zeta'',\ \xi'Y'z',\ X'y'\zeta',\ X'YZ'',$$
$$x''y'z,\ x''Y''\zeta'',\ \xi''\eta'\zeta,\ \xi''y''Z',\ X''\eta''z'',\ X''Y'Z\ ;$$

and the following 9 " Morley " triangles are *not* equilateral :

$$x\eta'\zeta'',\ \xi Y'Z'',\ Xy'z'',\ x'y''Z,\ \xi'\eta''z,\ X'Y''\zeta,\ x''Yz',\ \xi''y\zeta',\ X''\eta Z'.$$

In addition * we have

3 " x " triangles all equilateral—$xX'X''$, $\xi x'x''$, $X\xi'\xi''$,

3 " y " triangles all equilateral—$yY'Y''$, $\eta y'y''$, $Y\eta'\eta''$,

and 3 " z " triangles all equilateral—$zZ'Z''$, $\zeta z'z''$, $Z\zeta'\zeta''$.

W. J. D.

1177. Judges are conscientious when trying cases, and I always felt that in order to understand any motor case it was necessary that I work out the respective speed of each vehicle in a measure that would tell me their respective ground speeds. The only measure that would give me any mental picture of the speed at which a vehicle covered the ground was the measure of feet per second. That involved me in a lot of arithmetic. Sixty miles per hour works out at 87·9 recurring feet per second, and every time I converted miles per hour into feet per second I got a result in recurring decimals. So then I had to look for a simple formula, and this is how I got it. Instead of calling 60 m.p.h. 87 odd feet per second, I called it 90 f.p.s. and that gave me the simple formula of adding half to my miles per hour to obtain speed in feet per second correct within 2 per cent. Ever since then I have driven cars, and tried running down cases in feet per second. Now what I say to all motorists is that they try doing what I do, that is, always to drive and think of speed in feet per second instead of in miles per hour, and you will at once become a hundred per cent. better and safer driver. All you have to do is to add one half to the figure of your speed in m.p.h. and you will get your speed in feet per second. Any child can do that. The other aspect of road safety touches what is called kinetic energy, which means the moving force possessed by a vehicle in motion. I can't give you a more detailed explanation, but another way to put it is to refer to kinetic energy as the kick possessed by a moving vehicle. A small motor car weighing about a ton and moving at a speed of 40 miles per hour strikes the same blow as eighteen ten-ton steam rollers travelling at their highest speed, which is 3 miles per hour. That is the force you are handling when you speed up a light car to 40 miles per hour—60 feet per second. If you are driving a big seven-seater two-ton car at 60 miles per hour (90 feet per second) its kinetic energy is more than that of 100 ten-ton steam rollers moving at 3 miles per hour.—*The Listener*, October 1937, p. 892.

[Per Mr. R. M. Wright; Mr. F. Ayres; Mr. W. G. Faires.]

*I am indebted to Mr. S. W. Finn for drawing my attention to these 9 equilateral triangles. They are not, of course, " Morley " triangles, but they led to the discovery of the completed Fig. 8.

EUCLID.*

By W. C. Fletcher.

In this address I use the word Euclid not as the name of the man but as a short title for the XIII books of his *Elements of Geometry*. The "Euclid" of our schooldays was but a small part of the whole, just as the "Newton" of our Cambridge days was a mere fragment of the *Principia*. We learnt the whole of Books I-IV, about half of VI and about half of XI; of the rest we knew nothing.

I should like first to say a little about those other books. Books VII, VIII, IX deal with properties of numbers, *i.e.* integers, with H.C.F., L.C.M. and properties such as:

$$\text{if} \quad a:b=c:d, \quad \text{then} \quad a+b:a-b=c+d:c-d,$$

that is, with material to be found to-day in school algebras. They include the summation of geometrical progressions and conclude with a proposition showing how "perfect numbers" can be found. The material of these books accordingly is part of our ordinary working stock. Book X deals with the classification of irrationals of quadratic type and is happily rendered unnecessary to-day by algebraic expression.

I have said or implied that VII-IX contain the theory of ratio and proportion, but only so far as integers are concerned. This of course is quite enough for all but pure mathematicians; but "Euclid" is very emphatically a piece of Pure, or to use Klein's term, Precision mathematics, and the theory is extended to cover irrationals in Book V. It is curious that having given the rigid theory in V, Euclid should leave standing the limited, simpler theory in the later books. The propositions of course are the same; probably the simple theory was known and the resulting rules used before the discovery of incommensurables created a scandal among the "Greek intellectuals" of whom Professor Hogben speaks so contemptuously. Book V allayed the scandal, that is, it showed that the familiar rules were applicable even when the ratios concerned were those between incommensurable quantities. In our current school practice we quite rightly ignore the difficulty which Book V faces and meets; it is quite difficult enough for boys and girls to get hold of the idea of ratio and to use the rules of proportion confidently and accurately.

There remain the second half of XI, XII and XIII. Of these XIII contains the geometry of the five regular solids—very beautiful material for study, but its study is greatly simplified to-day by the use of simple algebra and trigonometry. Nor have these solids to us the special interest which they had to the Greeks and for centuries later. One of the oddities of history is that Kepler, the very man who later was to discover that the orbits of the planets were

* A paper to the London Branch of the Mathematical Association, 19th February, 1937.

ellipses, began life thinking that the Creator in designing the universe must necessarily have had His eye on the " perfect solids " and that He arranged the " spheres " of the five planets in accordance therewith.

Books XI (later half) and XII deal with the volumes of prisms, pyramids, cylinders, cones and spheres, that is, with what we include in Mensuration and treat, I fear, in a rather slovenly fashion. Here I think we might well make some use of his methods, particularly as they afford practice in the valuable art of drawing three-dimensional objects. He begins by proving that if a parallelepiped be cut in two by a plane parallel to the ends, the two pieces are in the ratio of their bases. This he proves precisely as he proves the two similar propositions VI, 1 and VI, 33, namely, that parallelograms of the same height are proportional to their bases and that angles at the centre of a circle are proportional to their arcs, that is, he uses his famous definition, or test as I should prefer to call it, of equality of ratios.

Then he proves that parallelepipeds of the same height and on the same base are equal. The proof is exactly like that of the fundamental proposition on the areas of parallelograms but involves two steps instead of one and a good piece of drawing. Then he proves that parallelepipeds of the same height are proportional to their bases and, finally, that similar parallelepipeds are in the triplicate ratio of their corresponding sides.

For circles, pyramids, cones and spheres he needs the " Principle of Exhaustion ", which he states thus (X, 1) : " Two unequal magnitudes being set out, if from the greater there be subtracted more than its half, and from the residue more than its half again and so on repeatedly, there will be left some magnitude less than the lesser magnitude set out." Thus to prove that circles are proportional to the squares on their diameters he inscribes squares, octagons, and so on, in each ; at each step this takes out more than half the residue from the previous step. The areas of the polygons are as the squares of their diameters. But to complete the argument, that is, to pass from polygons to circles, he has to use a tiresome process of *reductio ad absurdum*, and this he repeats in each case—pyramid, cone and sphere. In regard to this it is interesting to compare Newton's innovation in his first lemma and his own comment in the Scholium at the end of Section I : " Quantities which in any finite time tend constantly to equality and before the end of that time approach one another more nearly than by any assigned difference, *become ultimately equal*." " I have premised these lemmas to escape the tedium of making long reductions to absurdity in the manner of the ancient geometers."

The two interesting propositions in Euclid's treatment of pyramids are :

XII, 3. A triangular pyramid can be divided into two equal pyramids similar to the whole and into two equal prisms ; and the prisms are together greater than half the whole pyramid.

XII, 7. A triangular prism can be divided into three equal pyramids.

Both of these form excellent exercises in drawing and intuition, and the second a very fine exercise in the construction of the three pyramids from their nets.

In dealing with spheres he cannot use the exact form of the Method of Exhaustion quoted above and applied to circles and pyramids, but replaces it by XII, 17: "Given two concentric spheres it is possible to inscribe in the greater a polyhedral solid which does not touch the surface of the smaller." The solid in question is that which would result from planing off patches from a sphere in such a way that parallels of latitude would be reduced to regular polygons, that is, an approximate model of a sphere formed of isosceles trapezia ending in isosceles triangles at the poles —again a very valuable subject for construction in cardboard.

I think myself that this part of the subject which I have been discussing so far might afford much more interesting and profitable material for many of our pupils than much of that which we now set before them. However that may be, I pass now to what is nearer to current school practice.

The position in the schools to-day is this: we take for granted the fundamental propositions about congruent triangles and about parallels, and build all our subsequent work on that. When we are revising we start anywhere and work back, and find that everything in our geometry depends on one or other or on both of these fundamental ideas.

It is sometimes interesting to try to do by congruence alone what one is in the habit of doing by parallelism; parallelism often affords an easier but not the only way. As trivial examples I may suggest: if the opposite angles of a quadrilateral are equal, prove that the opposite sides are also equal; construct the tangent to a circle from an external point and prove your construction. The latter will lead you to Euclid's construction, which is not ours.

A more important example is: prove that the perpendiculars of a triangle intersect; and one which may lead you a pleasant and profitable dance is: prove that the medians intersect.

Well, we have, say, our Fifth Form boys, who have had a certain amount of geometrical training based, *via* the propositions specified above, on assumptions, and sooner or later the question arises, shall we go back and examine these propositions themselves or shall we go forward? If it is desirable to go back, I suggest that the first 29 propositions of Euclid afford a very suitable guide. I am not saying that we ought to go back; for many boys it may be more profitable simply to go on and learn some more substantial geometry. But taken at the stage we are considering—Fifth Form, not Third—these few early propositions need not take long nor be elaborately treated. In any case they should be taken backwards.

So I will run through these. Propositions 27-29 concern parallels,

and I will leave them till later and start with 26. This we deal with, if at all, by superposition, because we assume the angle sum of a triangle to be two right angles. This we cannot do now, for the proposition is independent of parallels. Euclid proves I, 26 (two angles and a side) by *reductio ad absurdum*, using I, 16—the exterior angle of a triangle is greater than either of the opposite interior angles. This I, 16 may be called a corner stone of his building, so great is its importance; we tend to lose sight of it by our early emphasis on the angle sum.

To prove I, 16 he bisects a side (I, 10), joins to the opposite corner,

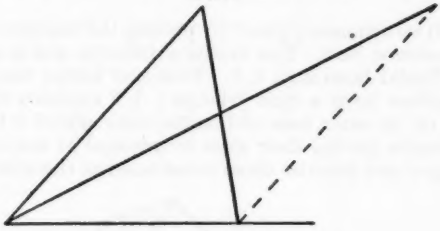


FIG. 1.

produces and cuts off an equal piece (I, 3). Vertically opposite angles are equal (I, 15), so we have congruent triangles (I, 4).

Consider I, 15; it depends on I, 13 that the adjacent angles are together equal to two right angles. Why does this come so late?

FIG. 2.

In current schoolbooks it is the first proposition. The question throws light on Euclid's methods. He includes a definition of a right angle with his others, prefixing them all to his propositions. But a definition does not necessarily establish the existence of the thing defined, and Euclid makes no use of right angles until he has proved that his definition describes a real entity. His constructions are in fact often existence theorems; and so it is in this case. To prove I, 13 he first has to erect a perpendicular; that is to say, he has to show that a line can be drawn so that the adjacent angles are equal. This is I, 11: cut off equal pieces on each side of the point (I, 3); describe an equilateral triangle (I, 1). Then the adjacent angles are equal (I, 8). To bisect a line (I, 10 used above) he describes an equilateral triangle (I, 1) and bisects the vertical angle (I, 9) and uses (I, 4). The construction for I, 9 is precisely

the same as for I, 11, though Euclid does not so regard it, for he does not recognise as an angle the angle between two lines in the same straight line.

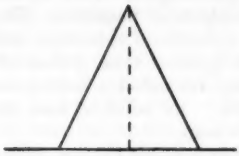


FIG. 3.

I, 8 (*SSS*) we commonly prove by putting the triangles on opposite sides of a common base. This evades a difficulty which is well worth facing, as Euclid faces it in I, 7. Everyone knows that three rods pinned together form a rigid triangle; I, 7 explains directly why this is so: on the same base and on the same side of it there cannot be two triangles having their sides terminated at one extremity of the base equal and likewise those terminated at the other extremity

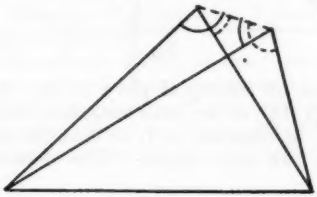


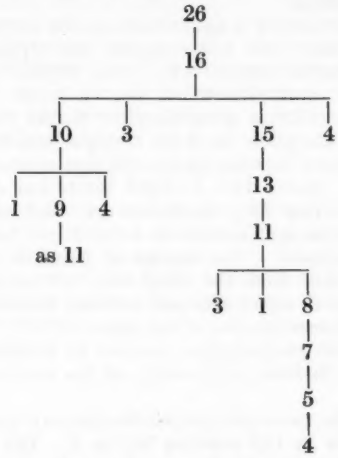
FIG. 4.

equal. The proof (like ours) depends on the equality of the angles at the base of an isosceles triangle (I, 5). It is easy to anyone who has had some training in geometry and in English, but was a terrible stumbling-block in former days. It is worth noting that though there are three cases Euclid gives only the one, as he often does, leaving it to his readers to fill in the gap.

I, 5 of course he proves by I, 4 (*SAS*), but since I, 13 is not yet in sight he has to prove at the same time that the external angles

FIG. 5.

are equal and has the well-known figure. Accordingly we have the following inverted genealogical tree :



So far everything is straightforward and easy ; if we wish to push the enquiry further, we get into deeper waters.

For instance : Euclid has proved (I, 11) that right angles can be constructed, *i.e.* that they exist—but are all right angles equal? This he can only answer by assuming it as a postulate. Again : how can we know whether lines are equal or not? Indeed, what does equality mean? The former question he answers by a postulate and two propositions. The postulate in question is that a circle can be drawn with any centre and any radius. This means, among other things, that he assumes that lines drawn from the same point can be equal and their equality recognised. Then in I, 2 and I, 3 he proves that, with this assumption, a line can be drawn from any point equal to a given line. The meaning and importance of these two propositions has, I think, been too rarely recognised.

We might go further and ask what is " distance " and what is " angle " ; but as Euclid raises neither of these questions I leave them to your own consideration.

Finally, something about parallels. Euclid proves (I, 27, 28) that under the familiar angle conditions certain lines will not meet ; so he calls these lines parallel according to his definition. What he cannot do with the assumptions already made is to prove that if the lines do not meet the angle conditions hold. Accordingly he lays down his axiom of parallels, establishes the angle relations and proceeds happily thereafter. This additional axiom was for centuries regarded as illegitimate. Proofs were put forward, but found illusory. At last in the nineteenth century is was recognised that

no proof was possible, and not only so, but also that the other axioms were not, as had commonly been supposed, indisputable truths but assumptions.

The real point for us now is not to take, as, for example, Sir Percy Nunn suggests, some less unacceptable assumption instead of Euclid's, but to realise clearly why some further assumption is necessary for the establishment of the euclidean system. The reason for this necessity is quite simple. Euclid has no effective definition either of the plane or of the straight line, and the modern definition of the plane depends on the still non-existent definition of the straight line. Now, with a slight limitation as regards the sphere, all Euclid's first 28 propositions are valid, not only in the plane, but also on the sphere and on a third less familiar surface, called the pseudosphere. The axiom of parallels is required to differentiate the plane from the other two. It completes, in fact, together with Euclid's other relevant axioms, the definition of the straight line and therewith also of the plane.

A series of simple propositions, obvious to intuition and easily proved in Euclid's fashion, goes nearly all the way to justify these assertions.

1. Let PL, QM be two equal perpendiculars to a line LM. Bisect LM at right angles by OA meeting PQ in A. The two halves of the figure are congruent, and the angles p and q are equal and OA bisects and is perpendicular to both lines.

2. If the perpendiculars from three points P, Q, R in a line to another are equal, the angles at P, Q and R are right angles, and Euclid's whole system follows.

3. Alternatively the third perpendicular may, so far as we know, be greater or less than the other two. In this case the locus of points equidistant from LM is not a straight line, and we have a curve called an equidistance locus (EQD) analogous in many respects to a circle. A straight line cannot cut it more than twice; it is everywhere at right angles to the perpendicular to the base.

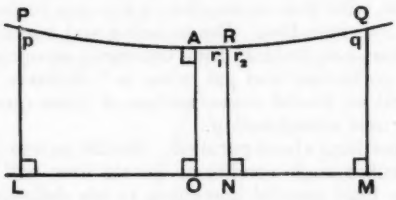


FIG. 6.

4. Now suppose $PL = QM > RN$.

Then since PQ cannot cut any EQD more than twice, the perpendicular increases continually as R recedes in either direction from A. By congruence and I, 16 the angles p, q, r_1 are acute, and

in general the fourth angle of every trirectangular quadrilateral is acute.

5. Also $r_1 > p$

and therefore $> q.$

(This is an extension of I, 16 to the quasi-triangle $RNMQ$.)

If we assume $RN > PL$, the inequalities in 4 and 5 are reversed.

6. Let ABC be any triangle. Bisect AB, AC in F, E; join FE and draw perpendiculars from A, B, C. Then, by congruence,

$$BK = AH = CL,$$

and the angle sum of the triangle is equal to the sum of the two equal angles B and C of the isosceles trirectangular quadrilateral $BKLC$, and the areas are also equal.

7. Let A move along an EQD based on EF; then since the quadrilateral does not alter, the angle sum and the area of the triangle remain constant.

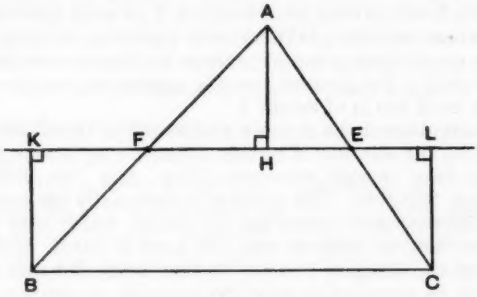


FIG. 7.

8. Just as Euclid's theory of parallels enables us to transform triangles without altering the area, this last proposition now gives us the same power. Hence two triangles can be transformed into a single one, whose area is the sum of the separate areas and in which the defect or excess of the angles below or above π is the sum of the separate defects or excesses.

9. Hence the defect or excess of the angle sum is proportional to the area.

10. This links on to the well-known property of the sphere and the corresponding property of the pseudosphere, and enables us to identify our two alternative suppositions with the phenomena of these surfaces. W. C. F.

1178. It commonly happens that when we ask silly questions, mathematical theory does not directly refuse to answer but gives us an oracular answer like 0/0 out of which we cannot wring any meaning.—Sir Arthur Eddington, *New Pathways in Science*, p. 102. [Per Mr. T. S. Venkatraman.]

SOME PROPERTIES OF POINTS ARRANGED AT RANDOM ON A MÖBIUS SURFACE.

By W. O. Kermack and A. G. McKendrick

(from the Laboratory of the Royal College of Physicians, Edinburgh).

If a series of points be pricked at random through a strip of paper, and the strip be held horizontally, certain of the points will appear as peaks, namely those which happen to be higher than either of their immediate neighbours. These we call maximal points. Similarly there are others, minimal points, which are lower than either of their immediate neighbours. Between each maximal point and an adjacent minimal point is a "run", either a "run up" or a "run down", the points in a run forming a steadily ascending (or descending) series. These runs will naturally be of various lengths, the length of a run being defined as the number of points in the run, including the first and the last. The shortest runs will thus be of length 2, others may be of length 3, 4, or some greater number, up to m, where m is the total number of points on the strip. Clearly where the points form a run of m there is only one run throughout the whole strip; the greatest possible number of runs is $m-1$, in which case each run is of length 2.

The distribution of the number and length of these runs depends in general on the solution of certain difference equations, and these have been fully treated elsewhere (*Proc. Roy. Soc. Edin.*, 1937, vol. lxii, pp. 332-376). The problem is essentially the same as that discussed fifty or sixty years ago by André, which may be stated as follows. Let us suppose that we have m cards, each marked with one of the integers 1 to m. In how many different ways can the m cards be arranged so that the sequence of numbers contains r runs, a run here denoting a steadily ascending or descending series of numbers? Following André we may denote the number of ways in which we obtain r runs by P_{mr}. It follows at once that as there are m ways of arranging the m cards, the chance of obtaining r runs in a single shuffle is $P_{mr}/m!$, and this we denote by p_{mr}. André investigated the value of P_{mr} for $r=1$, 2, $m-1$ and $m-2$, but he does not seem to have obtained the general solution. However, an expression for this has recently been given elsewhere (*loc. cit.*).

If we notice that the statistical problem depends only on the relations of inequality between the numbers, we see that the values of p_{mr} are entirely independent of the actual values of the numbers written on the cards, as long as no two of these are equal. With this limitation we may consider the numbers on the cards as being entirely arbitrary. It follows that p_{mr} is also the chance that our arbitrary points should give us m runs, for we may consider the heights of the points as corresponding to the values of the arbitrary numbers. It is not necessary to worry here about the condition of inequality, for the chance that any two arbitrarily chosen points

will be at equal heights is obviously vanishingly small and so does not sensibly affect p_{mr}.

We may now consider a slight variant of the above problem. Instead of arranging André's m cards in a single row, we may distribute them round a circular table. This corresponds to pricking our m points, not on a straight strip of paper, but on a cylinder obtained, for example, by gumming the ends of the strip together. We may denote by $\overline{p}_{mr}$ the chance of obtaining r runs in one of these cyclic arrangements of m numbers or points. It is easily seen that $\overline{p}_{mr}$ is zero for all odd values of r; for, as runs up and runs down alternate with each other, there must be equal numbers of each, and so the total number of runs must be even. Further, when m is even the maximum number of runs is m; when m is odd it is $m-1$. In both cases the minimum number of runs is two. These properties are in contrast to those of p_{mr}.

It may be shown (*loc. cit.*) that if $\overline{P}_{mr}$ represents the number of different cyclic arrangements of m unequal numbers which contain r runs, then the following difference equation holds:

$$\overline{P}_{m+1,\,r} = r\overline{P}_{mr} + (m-r+2)\,\overline{P}_{m,\,r-2} \quad (r \text{ even}). \quad \ldots\ldots\ldots(1)$$

It can be shown that this equation is satisfied if $\overline{P}_{mr}$ is the coefficient of $y^m t^r/m!$ in the expansion of $\bar{\phi} = \Phi(\tau e^{\lambda y})$ where $\lambda = \sqrt{1-t^2}$, $\tau = \dfrac{1-\lambda}{t}$ and Φ is an arbitrary function. If we write the arbitrary function as

$$A_0 + A_1\tau e^{\lambda y} + A_2(\tau e^{\lambda y})^2 + \ldots A_i(\tau e^{\lambda y})^i + \ldots\,, \quad \ldots\ldots\ldots\ldots(2)$$

then the coefficients A_i are to be chosen so as to satisfy the physical conditions of the problem. These may be stated as follows:

$$\overline{P}_{mr} = 0, \text{ for all odd values of } r \text{ as proved above.} \quad \ldots\ldots(3)$$

$$\overline{P}_{m,\,m+1} = 0 \text{ for } m \geqslant 1. \quad \ldots\ldots\ldots\ldots(4)$$

(This follows from the fact that there are never more than m runs in any arrangement of m numbers.)

Furthermore it is not difficult to prove that

$$\overline{P}_{m2} = 2^{m-2} (m \geqslant 2). \quad \ldots\ldots\ldots\ldots(5)$$

As τ is an odd function of t, any odd power of τ is also an odd function of t, and therefore generates values for $\overline{P}_{mr}$ (r odd). These, however, according to condition (3) are necessarily zero, whence it follows that A_i is zero when i is odd.

It may be shown that if we choose $A_i = \dfrac{2}{i}$, (i even) so that

$$\bar{\phi} = \Phi(\tau e^{\lambda y}) = -\log(1 - \tau^2 e^{2\lambda y}), \quad \ldots\ldots\ldots\ldots(6)$$

then conditions (4) and (5) are satisfied. Thus $\overline{P}_{mr}$ is the coefficient of $\dfrac{y^m t^r}{m!}$ in (6), and it follows that

$$\overline{P}_{mr} = \Sigma_s \frac{2}{s}(-)^{\theta+s} \left(\Delta^s_m C_{\frac{m}{2},\theta}\right) s^m, \quad \ldots\ldots\ldots\ldots(7)$$

where $\theta = \frac{r+s}{2}$ must be a positive integer, and $0 < s \leqslant r$. In fact s takes the values $r, r-2, r-4$, etc.

As the number of distinct cyclic arrangements of m numbers is $(m-1)!$, it follows that $\overline{p}_{mr}$, the chance in the cyclic problem of obtaining r runs with m numbers on cards or with m points on a cylinder, is $\overline{P}_{mr}/(m-1)!$. Thus we have the complete solution of the problem of the distribution of the number of runs contained in m points arranged round a cylinder.

Formula (7) however gives a value, not only for $\overline{P}_{mr}$ (r even), but also for $\overline{P}_{mr}$ (r odd), and this in turn gives corresponding values for $\overline{p}_{mr}$ (r odd). The generating function $\bar{\phi}$ may be written as

$$\bar{\phi} = -\log(1-\tau e^{\lambda y}) - \log(1+\tau e^{\lambda y}), \quad \ldots\ldots(8)$$

the odd powers of t cancelling, but a generating function for these odd terms is obtained by taking

$$\bar{\phi} = -\log(1-\tau e^{\lambda y}) + \log(1+\tau e^{\lambda y}). \quad \ldots\ldots(9)$$

These two functions are formed out of the odd and the even terms respectively of the function

$$\bar{\phi} = -2\log(1-\tau e^{\lambda y}). \quad \ldots\ldots(10)$$

As we know that the even terms have a physical meaning it is natural to enquire whether the odd terms are related to some analogous problem of runs on a surface.

The peculiarity of the problem is that even sets of runs are to be excluded. We must therefore find a system relating exclusively to odd sets of runs and yet presumably closely connected with the cyclic case of the points on the cylinder. A natural conjecture is that the problem in question may relate, not to a simple cylinder formed in the ordinary way from a strip of paper, but to a Möbius band made by twisting the strip through 180° before joining the ends together. If such a strip be made and a number of holes be pricked through it at random, it is easily verified that in going round the strip once we always encounter an odd number of runs. [Thus if 2 points are pricked, it will be found that they constitute a single run of 3 (the same point beginning and ending the run); if 3 points are pricked we obtain either a single run of 4, or three runs of 2.] This curious result, which is best seen from simple models, is dependent on the fact that the surface has only one edge, so that after going round the circuit once we come back to the original point, but it now appears to have moved transversely across to a position symmetrical with regard to the line down the middle of the band. As the band has only one surface as well as one edge, we find ourselves, after one complete circuit, looking at the same hole from the opposite side of the strip.

That this conjecture is in all probability true may be demonstrated experimentally—that is to say, by arranging sets of perhaps

five points on a Möbius band and finding out, by counting, the fractions of these which contain one, three and five runs respectively. This may be done conveniently by means of telephone numbers taken from a directory and imagining the points to be located on the band according to these numbers (*loc. cit.*). The main object of this note is to indicate how this conjecture may be proved.

As a single circuit round the band results in the interchange of the two edges, it is not convenient to regard the points as characterised by their distance from either edge, but rather to look on them as measured from the central line running along the strip. We may imagine the points to be added to the strip in the order of their distances from this line, first the nearest, then the next nearest, and so on, the one closest to either edge being added last. It is to be understood that the position along the band where the point is placed is decided by chance. In consequence of the ambiguity of sign which results from the fact that after a single circuit a point originally above the central line appears to be below this line, we regard the point as having equal chances of appearing at either of the two possible equidistant positions on the two sides of the central line. $\overline{p}_{mr}$ will be the chance that, when m points have been added, the strip will contain r runs. Now it is easy to see that the addition of an extra point will either leave the number of runs unchanged or increase it by 2. Let e_{mr} be the chance that the first possibility, and e'_{mr} be the chance that the second, will actually happen; so that

$$e_{mr}+e'_{mr}=1. \qquad \text{.....(11)}$$

Now any band with m points and r runs is to be regarded as uniquely formed by the addition of an extra point (that is to say, the one most distant from the central line) to a band with $m-1$ points which contains either r or $r-2$ runs. Thus we have the following equation:

$$\overline{p}_{m+1,\,r}=e_{mr}\overline{p}_{mr}+e'_{m,\,r-2}\overline{p}_{m,\,r-2}. \qquad \text{.....(12)}$$

Now e_{mr} is the number of ways of adding the extra point without change in r, divided by the total number of ways of adding this point. The latter number is equal to $2m$, because the new point may be placed in any one of the m spaces, between the previous m points, and in either of two ways within each space (that is, on either side of the central line). The number of runs will not be changed when the point is added on either side of a maximal or a minimal point provided that it falls on the same side of the central line as the maximal or minimal point. But the number of maximal plus the number of minimal points is equal to r, so that there are $2r$ ways of adding the point without altering the number of runs.

Whence
$$e_{mr}=\frac{2r}{2m}=\frac{r}{m}. \qquad \text{.....(13)}$$

So that $$\overline{p}_{m+1,\,r}=\frac{r}{m}\overline{p}_{mr}+\left(1-\frac{r-2}{m}\right)\overline{p}_{m,\,r-2},$$

or if we write $\overline{P}_{mr}=\overline{p}_{mr}(m-1)!$ (r odd),

$$\overline{P}_{m+1,\,r}=r\overline{P}_{mr}+(m-r+2)\,\overline{P}_{m,\,r-2}. \quad (r \text{ odd}) \ldots\ldots\ldots(14)$$

Thus the difference equation is the same as that for $\overline{P}_{mr}$ (r even), which occurs in the simple cyclic problem (1); and the general form of the generating function of $\overline{P}_{mr}/m!$ must be of the form $\Phi(\tau e^{\lambda y})$. The arbitrary function must be determined in such a way as to conform to the "boundary conditions" satisfied by $\overline{p}_{mr}$ (r odd).

As above, these may be stated as follows:

$$\overline{P}_{mr}=0 \ (r \text{ even}), \ldots\ldots\ldots(15)$$

$$\overline{P}_{m,\,m+1}=0 \ (m \geqslant 1), \ldots\ldots\ldots(16)$$

$$\overline{P}_{m1}=1. \ldots\ldots\ldots(17)$$

Condition (17) follows from the equation $\overline{P}_{m+1,\,1}=\overline{P}_{m1}$ (obtained by putting $r=1$ in (14) and noting that $\overline{P}_{2,\,1}=(2-1)!\,\overline{p}_{21}=1$). Now if we put

$$\overline{\phi}_1=-\log(1-\tau e^{\lambda y})+\log(1+\tau e^{\lambda y}), \ldots\ldots\ldots(18)$$

we observe that $\overline{\phi}_1$ contains no even powers of τ, and therefore no even powers of t, so that $\overline{P}_{mr}=0$ (r even). Also by (18) $\overline{P}_{m1}=1$; and in general

$$\overline{P}_{mr}=\sum_s \frac{2}{s}(-)^{\theta+s}\left(\Delta^s_m\, C_{\frac{m}{2},\,\theta}\right)s^m \ (r \text{ odd}), \ldots\ldots\ldots(19)$$

$$=\sum_s \frac{1}{2s}K^s_{m+3,\,r}\,s^m,$$

(where K^s_{mr} has the value $4(-)^{\theta+s}\Delta^s_m\, C_{\frac{m-3}{2},\,\theta}$, *loc. cit.*) ……(20)

$$=2\sum_s K^s_{m'r}s^{m'-4}. \ldots\ldots\ldots(21)$$

By making use of the properties of K^s_{mr} already established (*loc. cit.*) it can be shown that

$$\sum_s K^s_{m'r}s^{m'-4}=0 \ldots\ldots\ldots(22)$$

when $r \geqslant m'-2$, that is when $r \geqslant m+1$.

Thus $$\overline{P}_{m,\,m+1}=0,$$

and so condition (16) is satisfied.

It follows that

$$\overline{p}_{mr}\left(=\frac{\overline{P}_{mr}}{(m-1!)}\right)$$

is the chance that m points on a Möbius band should give r runs.

In conclusion we may observe that the problem of calculating the distribution of the lengths of the individual runs on a Möbius band does not differ essentially from the analogous problem for the cylinder. The latter case has already been discussed fully (*loc. cit.*), and it has been shown that if $\bar{x}^m_{nr}$ is the chance that any r consecutive runs taken at random on a cylinder containing m points, should contain n points then, provided that $m > n+1$, $\bar{x}^m_{nr}$ is independent of m and is equal to $\frac{3}{2}\Delta^2_n f_{nr}$, where

$$f_{nr} = \frac{1}{n!}\sum_s K^s_{nr}\, s^n, \quad K^s_{nr} \qquad \text{.....(23)}$$

having the value already given (20). Exactly the same formulae apply in the case of the Möbius band. Thus the chance that any single run is of length n is obtained by putting $r=1$, and is given by

$$\bar{x}^m_{n1} = 3\,\Delta^2 \frac{1}{n!} \qquad \text{.....(24)}$$

whilst the chance that 2 consecutive runs (a " gap ") should contain n points is

$$\bar{x}^m_{n2} = \frac{3}{2}\Delta^2 \frac{2^n}{n!} \qquad \text{.....(25)}$$

For convenience the numerical values of $\bar{P}_{mr}$ for different values of r, and of $\bar{x}^m_{n1}$ and $\bar{x}^m_{n2}$ for different values of n are given below.

The values of $\bar{P}_{mr}$ are conveniently calculated by means of the difference equations (1) and (14); they may also be obtained from the general formulae (7) and (19). Formula (19), for example, gives us

$$\left.\begin{aligned} \bar{P}_{m1} &= 1^m, \\ \bar{P}_{m3} &= \frac{3^m}{3\,.\,2^2} - \frac{(2m-1)}{2^2}\,1^m \qquad \text{.....} \\ \bar{P}_{m5} &= \frac{5^m}{5\,.\,2^4} - \frac{2m-3}{3\,.\,2^4}\,3^m + \frac{m^2-3m+1}{2^3}\,1^m. \end{aligned}\right\} \quad (26)$$

These formulae incidentally illustrate the type of expression obtained from (7) and (19) for particular values of r.

I. *Cylinder.* Values of $\bar{P}_{mr}$ (r even).

	$r=2$	$r=4$	$r=6$	$r=8$	$r=10$	$(m-1)!$
$m=$ 2	1					1
3	2					2
4	4	2				6
5	8	16				24
6	16	88	16			120
7	32	416	272			720
8	64	1824	2880	272		5040
9	128	7680	24576	7936		40320
10	256	31616	185856	137216	7936	362880

II. *Möbius band.* Values of $\overline{P}_{mr}$ (r odd).

	$r=1$	$r=3$	$r=5$	$r=7$	$r=9$	$(m-1)!$
$m=2$	1					1
3	1	1				2
4	1	5				6
5	1	18	5			24
6	1	58	61			120
7	1	179	479	61		720
8	1	543	3111	1385		5040
9	1	1636	18270	19028	1385	40320
10	1	4916	101166	206276	50521	362880

It is to be noted that $\Sigma_r \overline{P}_{mr}$ (r odd) $= \Sigma_r \overline{P}_{mr}$ (r even) $= (m-1)!$ This follows from the fact that $\Sigma_r \overline{p}_{mr}$ (r odd) and $\Sigma_r \overline{p}_{mr}$ (r even) are both necessarily equal to unity in virtue of the physical nature of the problem. The result offers a convenient check of the correctness of the numbers in the tables.

III. Values of $\bar{x}^{m}_{n1} = 3\,\dfrac{(n^2+n-1)}{(n+2)!}$.

$n=2$	5/8	0·6250000
$=3$	11/40	0·2750000
$=4$	19/240	0·0791667
$=5$	29/1680	0·0172619
$=6$	41/13440	0·0030506
$=7$	55/120960	0·0004547
$=8$	71/1209600	0·0000587
$=9$	89/13305600	0·0000067
$=10$	109/279417600	0·0000004
		0·9999997

IV. Values of $\bar{x}^{m}_{n2} = 3\,.\,\dfrac{2^{n-1}}{n!}\,.\,\dfrac{n-2}{n+2}$.

$n=3$	2/5	0·4000000
$=4$	1/3	0·3333333
$=5$	6/35	0·1714286
$=6$	1/15	0·0666667
$=7$	4/189	0·0211640
$=8$	3/525	0·0057143
$=9$	2/1485	0·0013468
$=10$	4/19175	0·0002086
		0·9998623

W. O. K.
A. G. M'K.

NEW GEOMETRY FOR GERMANY.

F. B. PIDDUCK; E. BOREL; D. H. SADLER.

THE circle will in future have 400 degrees in Germany, instead of 360, as at present, and the right angle 100, instead of 90. This is ordered in a new decree of the German Minister of the Interior.

Degrees will continue to be divided into minutes and seconds, but these will also be arranged on the decimal principle.

Several years will be allowed to elapse before the new geometry is made obligatory, especially for use at sea, but it will be taught in the schools next year.—*Daily Telegraph and Morning Post*, Nov. 1, 1937.

It is announced in the press that the right angle will be divided in future into 100 parts in German instruction, and the degree and second correspondingly. It is well known that this step was taken during the French Revolution and given up presumably owing to the labour of translating tables into a new notation. Is it too late? It is always easy to say that the labour is prohibitive, but surely it would be an easy matter to prepare tables sufficient for practical use for the present. The practical advantage of degrees and decimals is too great to be overlooked or to be stifled by inertia. What we would like to see is a wide discussion of this matter, followed by action by the Board of Education. F. B. PIDDUCK.

La situation actuelle en France est la suivante, relativement à la question de la mesure des angles.

L'ancienne division de l'angle droit en 90° est utilisée par les astronomes et par les marins ; par contre, la division en 100 grades est utilisée par les géodésiens et notamment par le Service géographique de l'armée, qui publie les meilleures cartes de la France. Je pense que cette situation restera stationnaire au point de vue national aussi longtemps que la question ne sera pas étudiée internationalement, car les astronomes et les marins de tous les pays ont intérêt à utiliser les mêmes notations. Je pense, d'autre part, que si plusieurs autres pays, dans des congrès internationaux, proposaient que la division centésimale devienne universelle, les savants français soutiendraient ce point de vue, car ce sont les fondateurs français du système métrique qui ont les premiers proposé cette division centésimale. On a déjà fait remarquer que la plus grande difficulté proviendrait de nos habitudes pour la division du temps ; on a observé également que notre journée comprend 96 quarts d'heure et que par suite, si l'on la divisait en 100 parties égales, chacune des divisions, pour laquelle on pourrait créer un nom nouveau, différerait fort peu du quart d'heure. Seulement, cela remettrait en question toutes les unités du système électrique C.G.S., qui sont basées sur la seconde sexagésimale. E. BOREL.

A report that the old-established division of the right angle into 90 degrees is to be superseded in Germany by a centesimal division

into grades, seems to have had considerable publicity in the press. In spite of the startling headline, which is reproduced above, the news paragraphs are rather restrained in that they state that the edict is to apply immediately only to the teaching of the new system; we are led to believe, however, that in a few years the use of the centesimal division of the quadrant will be obligatory. In view of the far-reaching effects of such a change it is timely to examine both the authority of the report and the possibility of such a geometrical revolution in Great Britain.

Enquiry from leading German mathematicians and astronomers has revealed that the reports have been greatly exaggerated; it is impossible to believe that any decree involving such a great revision of mathematical and astronomical tables, instruments and records, to say nothing of the confusion bound to arise in minds inured to the old system, could have been promulgated without the *knowledge* of the leading authorities in mathematics and astronomy. What importance and what weight are we then to attach to such a report? It would seem probable that an attempt is being made to introduce the centesimal system gradually, (but not by degrees!), in the first place by teaching the new system in schools, by using it wherever possible for theoretical calculations and possibly by extending it to geodetic work. That is exactly the extent to which this system has progressed in France (the home of the metric system) after 150 years. As pointed out by Professor Borel in his summary of the position in France, a further extension to cover the two main uses in astronomy and navigation demands international agreement. There must only be a small chance of this, and even if the principle were approved, how many generations would be necessary before the new system could be safely brought into universal use?

For navigation, it is only necessary to mention charts, sextants, compasses (both magnetic and gyro), navigational tables, tabulated bearings and great circle courses, to realise the huge capital cost and enormous work involved in making any such change. A sudden change would mean years of preparation, with a great wastage of instruments and tables, while a gradual change would involve probably even more duplication, together with confusion in a subject where a mistake can sink a ship. Over and above the practical aspect, there is the consideration that, in general, navigators are not trained mathematicians to whom such a change of unit is merely a matter of conversion; working difficult problems in conditions, which at the best would upset the supposedly temperamental mathematician, and which at the worst would render him helpless, calls for intense simplification and the obviation of the necessity for great judgment or thought. Navigators are notoriously conservative, but such an attitude is quite easy to understand. The introduction of a new system of angular measurement, if sudden, would necessitate the revision of methods and formulae, which had been learned slowly and practised continuously, while a gradual change would leave these navigators (and there are many), who

work entirely by rule of thumb, between the Scylla and Charybdis of grade and degree; there would undoubtedly be the most determined opposition.

To astronomers, it is hoped that the more personal aspect, involving only temporary inconvenience and care, would have less weight, but the practical difficulties are almost if not as great. The division of circles for astronomical instruments is a costly and intricate work; astronomical tables, ephemerides, star catalogues and observatory records run into thousands of volumes and their reproduction or duplication would be an almost impossible task. The simple solution of conversion tables, involving extra work that would be shared by all, would be the only easy way of changing over; but would any generation deliberately hamper its work for the benefit of its successors?

The mathematician or the theoretical worker, on the other hand, can work with equal facility in any system he wishes, and provided that his modest tabular requirements (in this case mainly trigonometrical tables) are available, he suffers no inconvenience, and in fact gains from the undoubted merits of the decimal division. There may not be the same selection of tables to choose from at present, but to four or five figures there must be many tables published in France and the continent, while such tables as Peters (six-figure) and Roussilhe and Brandicourt (eight-figure) would seem to serve a considerable proportion of the more elaborate requirements. Should there be, however, any prospect of an immediate change to the centesimal system and therefore of an increased demand for such tables, it is quite certain that the compilers of mathematical tables would be well ahead of public demand, even if tables to a large number of figures at a small interval did have to await subsidised publication. This is well illustrated by the present tables of natural trigonometrical functions, which were mainly published long before the recent great increase in mechanical computation.

For theoretical astronomical work, the sexagesimal division of the degree has been superseded to a large extent by the decimal division of the degree, and although there is not a complete set of suitable tables available (probably owing to the small demand of a limited class), there are sufficient for most purposes. There is very little to choose in practice between this system and that of the centesimal division of the quadrant, but unless the latter system can be adopted universally, the decimal division of the degree is far superior, when comparisons have to be made with observations or results in sexagesimal measure.

The difficulties of and objections to a change of system of angular measurement have been outlined for the two classes of users likely to be affected most by such a change. The question arises as to whether these difficulties are insuperable or not; there can be no question that the division of the quadrant into 100 grades is fundamentally superior to its division into 90 degrees. An exact analogy

with the British monetary system would be erroneous, as so many different interests are involved, but in one respect, at least, the problems to be solved are the same—namely, that temporary convenience must be sacrificed to attain a much superior system. In 1853 a Select Committee was appointed to take into consideration and to report to Parliament the practicability and advantages, or otherwise, that would arise from adopting a decimal system of coinage. After discussing the alterations to the existing coinage and making definite recommendations for the details of the changes that would be necessary, "They [the Committee] believe that the necessary inconvenience attending a transition state will be far more than compensated by the great and permanent benefits which the change will confer on the Public of this country, and of which the advantages will be participated in to a still greater extent by future generations." De Morgan in his *Budget of Paradoxes* * gives a characteristically racy and extremely interesting account of the manner in which this report was debated upon in the House. It is perhaps not to be wondered at, that eighty years after the above report was written we were recently paying income-tax at the rate of 4s. 9d. in the pound, instead of at, say, 0·23 or 0·24 pounds. It is only in recent years that the difficulty (one might almost say practical impossibility) of constructing calculating machines to multiply and divide in units of the present currency, has resulted in the decimalisation of the pound as an intermediate step in extensions, and latterly to the publication of accounts in pounds and decimals of a pound. In this case the advantages are so great and obvious, that a huge class of users convert to decimals and back again to shillings and pence, and yet there seems no immediate prospect of a change being made.

While the change in the monetary system affects everyone to a larger or smaller extent, the change of angular measure affects a minority only, but means to that minority proportionally a far greater reconstruction. There are thus two main points of difference between the two proposed changes: the one relatively favouring the change in angular measurement, since the minority which it affects are presumably capable of understanding the proposal and of viewing it dispassionately, the other relatively opposing it by the practical difficulties attendant on the change. Can we hope for or expect any better progress than for the decimalisation of the coinage?

D. H. SADLER.

CORRESPONDENCE.

"A WEIGHTY MATTER".

To the Editor of *The Mathematical Gazette*.

DEAR SIR,—Readers of the cobweb-clearing correspondence on the miseries of mass and weight may be interested in the attitude towards these qualities in aeronautical engineering. So long as engin-

* Edition of 1872, page 362 *et seq.* Edition of 1915, Vol. II, page 169 *et seq.*

eering structures were either fixed, or had no vertical accelerations and only small ones in the horizontal plane, the muddle was not so very serious. The aircraft engineer cannot tolerate confusion, for only when the aircraft is in steady level flight do the inertia forces and the weight coincide.

In aeronautical engineering the fundamentals are the directly perceived qualities of length, time, and force: in British units, the foot, second, and pound (weight). Masses are very properly derived from the definition $m = P/f$. Hence the unit of mass is the Slug. For some reason such few textbooks as refer to the Slug at all refer to it contemptuously, and cling to the length-time-mass system that has the Poundal as the unit of force. The Poundal does not exist outside the examination room. Probably it never did.

Forces are considered as arising from the accelerations of masses, and the force due to gravitation is no exception. "g" is definitely an acceleration, not a mysterious divisor for converting weights into masses. Thus an instrument for the measurement of vertical accelerations would read "$1g$" in an aircraft in steady level flight, and "0g" (zero) if falling freely in a vacuum. The only way in which "g" keeps its hierarchal status above other accelerations is that it is usual to quote accelerations in terms of it, for direct comparison with the forces on the structure (or the human body) when at rest on the ground.

It may be mentioned also that numerical factors and parameters are always made non-dimensional.

Thus the student who enters aeronautical engineering will have to learn, in spite of all textbooks, what your correspondents have suggested he should be taught to begin with. If you are doing the job every day, there seems to be no difficulty. You would be as likely to confuse a space and a time because they had the same numerical measure in some units. But beginners do find it puzzling. They have to throw out of their heads abominations like centrifugal forces given by Wv^2/gr, which suggests either that gravity affects the radial acceleration, or that mathematicians are like the White Knight who

> "was thinking of a plan
> To dye one's whiskers green,
> And always use so large a fan
> That they could not be seen."

The problem would cause no difficulty if, at the very beginning of mechanics, pupils were taught that force, length and time are the most direct of mechanical properties, and later that mass is defined by a force producing a certain acceleration.

Every day I pass through an office upon the wall of which some defeated stressman has pinned the following couplet:

> "When the wind is in the West
> To divide by g is best."

Yours truly,

R. A. FAIRTHORNE.

MATHEMATICAL NOTES.

1277. *Notation for the quarterperiods of Jacobian elliptic functions.*

For many years now, in lecturing on the Jacobian elliptic functions I have replaced the classical K, iK' by K_c, K_n, written K_d for $-(K_c+K_n)$, and used K_s as a symbol for 0. With this notation, the function pq u is the function with simple zeros congruent with K_p, simple poles congruent with K_q, and unity for leading coefficient at the origin: Glaisher's notation is a morphological notation instead of pure shorthand. The difference $K_p - K_q$ is a halfperiod, and therefore, if p, q, r are c, d, n in any order, the functions sp u, ps u, qr u have K_p for a halfperiod and K_q, K_r for quarterperiods.

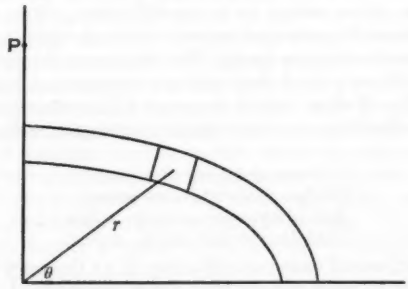
Needless to say, results have to be proved, but there is practically nothing to be memorised and remembered, and this is what matters to the learner and to the middleaged. E. H. N.

1278. *Attraction of spheroids.*

The gravitational potential of a shell bounded by two similar oblate spheroids at a point on its axis is

$$\frac{M}{ae}\tan^{-1}\frac{ae}{z},$$

where M is the mass, a the major axis, e the eccentricity, and z the distance of the point from the centre of the spheroid. This result appears to be new and is easily proved by using the eccentrie angle and manipulating the result of integration.



The element of volume is

$$\begin{aligned} &dp \,.\, ds \,.\, r\cos\theta\, d\psi \\ &= kp\, ds \,.\, r\cos\theta\, d\psi \quad \text{(since the spheroids are similar)} \\ &= kr^2 d\theta \,.\, r\cos\theta\, d\psi. \end{aligned}$$

$$V = \int_0^{2\pi} d\psi \int_{-\pi/2}^{\pi/2} \frac{kr^2\, d\theta \,.\, r\cos\theta}{R},$$

where $$R^2 = z^2 - 2zr\sin\theta + r^2.$$

Now $$r\cos\theta = a\cos\phi, \quad r\sin\theta = b\sin\phi,$$

$$\tan\theta = (b/a)\tan\phi,$$

$$\sec^2\theta\, d\theta = (b/a)\sec^2\phi\, d\phi.$$

Thus $$V = 2\pi\int_{-\pi/2}^{\pi/2} \frac{ka^2b\cos\phi\, d\phi}{\{z^2 - 2bz\sin\phi + a^2 - a^2e^2\sin^2\phi\}^{\frac{1}{2}}}.$$

On integration this becomes

$$2\pi ka^2b\{\sin^{-1}\alpha - \sin^{-1}\beta\}$$

where $$\alpha = \frac{bz + a^2e^2}{a\sqrt{(z^2 + a^2e^2)}}, \quad \beta = \frac{bz - a^2e^2}{a\sqrt{(z^2 + a^2e^2)}}.$$

After a little manipulation this becomes

$$4\pi ka^2b\tan^{-1}\frac{ae}{z}.$$

Expanding $\tan^{-1}(ae/z)$, we have for a point on the axis of the spheroid

$$V = M\left\{\frac{1}{z} - \frac{1}{3}\frac{a^2e^2}{z^3} + \frac{1}{5}\frac{a^4e^4}{z^5} - \ldots\right\},$$

and at any outside point (r, θ)

$$V = M\left\{\frac{1}{r} - \frac{1}{3}\frac{a^2e^2}{r^3}P_2(\cos\theta) + \frac{1}{5}\frac{a^4e^4}{r^5}P_4(\cos\theta)\ldots\right\}.$$

Putting $M = 4\pi a^2e\, da$ and integrating with respect to a, the potential of a solid spheroid is found. F. W. Dyson.

1279. *Steiner's theorems on incentres and ecentres.*

1. The sixteen incentres and ecentres of the four triangles formed by four lines lie by fours on eight circles; four of these are coaxal, the other four are also coaxal and orthogonal to the first four; the two radical axes meet in the Wallace point, the point of concurrence of the four circumcircles of the four triangles.

These theorems were enunciated by Steiner * without proof. A proof with a very complicated figure was given by Mention †; this was simplified by Clawson,‡ and, so far as I know, these are the only places where an elementary proof is to be found. (The theorems can be shown also from the properties of cubic curves.) Unfortunately Clawson's notation is rather complicated, and a more convenient one arises when the lines are oriented.

* *Ges. Werke*, I, p. 223.

† *Nouv. Annales de Math.* (1862).

‡ *Annals of Mathematics*, 20 (1919).

2. Orient the lines in the figure so that directions BC, CB', $B'C'$, $C'B$ go the same way round the quadrilateral $BCB'C'$. The bisectors drawn bisect the angles formed by the directed lines *both from* a common vertex. Denote these bisectors from A, A', B, B', C, C' respectively by a, a', b, b', c, c'. Denote the other bisectors by a_1, a_1', b_1, ...

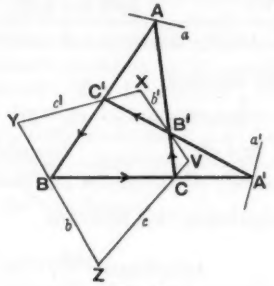


FIG. 1.

It is very easy to prove that $XYZV$ is cyclic ; denote its circumcircle by $bcb'c'$, the notation reminding us of the bisectors forming the sides of the quadrilateral. This circle goes through the points bc, cb', $b'c'$ and $c'b$. Similarly, the points $c_1'b_1'$, $b_1'c_1$, c_1b_1 and b_1c_1' are concyclic. Denote their circle by $b_1c_1'b_1'c_1$.

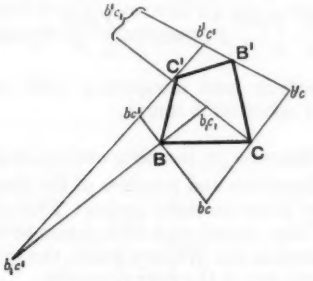


FIG. 2.

In this way we get, in all, eight circles, which we separate into two sets :

(i) $bcb'c'$, $b_1cb_1'c'$, $bc_1b'c_1'$, $b_1c_1b_1'c_1'$;

(ii) $b_1c_1b'c'$, $bc_1b_1'c'$, $b_1cb'c_1'$, $bcb_1'c_1'$.

Now the circles of the first set cut those of the second set orthogonally. For let O_1, O_2 be the centres of $bcb'c'$ and $b_1c_1b'c'$; then using *crosses*

$$\measuredangle(O_1 \,.\, b'c' \,.\, b'c) = -(b'c' \,.\, bc \,.\, b'c),$$
$$\measuredangle(O \,.\, b'c' \,.\, b'c_1) = -(b'c' \,.\, b_1c_1 \,.\, b'c_1).$$

But bc, $b'c'$, b_1c_1 lie on bisector a_1. Hence the difference between the crosses is the cross between the join of b_1c_1 to $b'c_1$ and the join of bc to $b'c$, that is, between c_1 and c, which is a right cross. Hence the orthogonal cutting follows.

3. The diagonal triangle of the quadrangle $bc \,.\, b'c \,.\, b'c' \,.\, bc'$ is $a_1a_1' \,.\, bb' \,.\, cc'$. (the quadrangle is $XYZV$ of Fig. 1). Hence a_1a_1' is the pole of the line $bb' \,.\, cc'$ for the circle $bcb'c'$. Now the triangles abc, $a'b'c'$ are in perspective, for the join of the vertices bc and $b'c'$

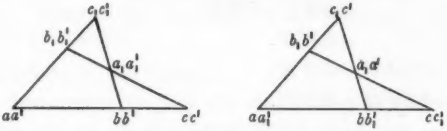


FIG. 3

is a_1, and similarly the joins of the other corresponding vertices are b_1 and c_1, while a_1, b_1, c_1 concur. Hence the point aa' is on the line $bb' \,.\, cc'$. Hence aa' and a_1a_1' are conjugate points for the circle $bcb'c'$. Similarly bb' and b_1b_1', cc' and c_1c_1' are conjugate pairs for all circles of set (i), and aa_1', a_1a' and bb_1', b_1b' and cc_1', c_1c' are conjugate pairs for all circles of set (ii). Thence the circles on diameters $aa' - a_1a_1'$, $bb' - b_1b_1'$, $cc' - c_1c_1'$ cut the circles of set (i) orthogonally, and those on diameters $aa_1' - a_1a'$, $bb_1' - b_1b'$, $cc_1' - c_1c'$ cut those of set (ii) orthogonally. But the circles of set (i) are orthogonal to those of set (ii), whence the two quadrilaterals, whose opposite vertices are the ends of these diameters, are so situated that the diameter of each is the line of orthocentres of the other, and these lines are the radial axes of the circles of sets (i) and (ii).

4. It remains to show that these lines go through W. Now bb', b_1b', bb_1', b_1b_1' are orthocentric points (for example, b is perpendicular to b_1). Hence their nine-point circle goes through B, B' and L, M, the mid-points of $bb' - b_1b_1'$ and $b_1b' - bb_1'$; and LM is a diameter of that circle. But L, M are on the diameters of the first and second quadrilaterals respectively, mentioned at the end of the last section, and as these diameters are perpendicular they meet on the nine-point circle, at F, say.

Hence $$\angle BFB' = BLB'.$$

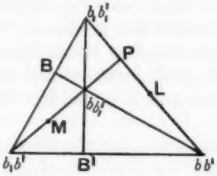


FIG. 4

Let W be the Wallace point of our original four lines. Now B, B', bb', b_1b_1' lie on a circle, centre L. Hence

$$\begin{aligned}\sphericalangle BLB' &= 2\sphericalangle (B\,.\,bb'\,.\,B')\\ &= 2\sphericalangle (bb'\,.\,BC + BCB' + CB'\,.\,bb')\\ &= \sphericalangle ABC + 2BCB' + CB'A'\\ &= ABC + BCA + A'CB' + CB'A'\\ &= BAC + CA'B'\\ &= BWC + CWB'\\ &= BWB'.\end{aligned}$$

Hence $\sphericalangle BLB' = BWB'$.

Hence $\sphericalangle BFB' = BWB'$.

So F is on the circle $BB'W$ and similarly on the circles $AA'W$, $CC'W$, but as these are not coaxal, it follows that F and W coincide.

H. G. FORDER.

1280. *A question in conics.*

1. I am indebted to a correspondent for the following question, which, I understand, was taken from a " Mays " paper.

If P is a fixed point in the plane of a central conic σ, then the envelope of the polar line of P with respect to conics confocal with σ is a parabola ω whose directrix is the join of P to the centre of σ. Also (i) if Q is the focus of ω, then P, Q are symmetrically related; (ii) the parabola ω is the envelope of the principal axes of all conics touching the tangents common to any circle κ with centre P and any conic confocal with σ.

The first part is well-known bookwork, and we may add that ω touches the principal axes of σ. (i) is quite easy, depending on the existence of the orthocentre of a triangle; the sting of the question is in (ii).

To show (ii). If AA', BB', CC' be pairs of opposite vertices of the quadrilateral of common tangents to the circle κ and a fixed conic γ confocal with σ, then by a well known theorem of Chasles these pairs are each on a conic confocal with σ, and the join of a pair is the polar of P for the conic through the pair. Now the envelope of the polars of P for conics touching the four common tangents is a conic, say ϵ, and as one of the polars is that for the circle, and so is the line at infinity, it follows that ϵ is a parabola; three other conics touching the four lines are the degenerate point-pair conics (A, A'), (B, B'), (C, C'), and the polars of P for these are the lines AA', BB', CC'. Hence ϵ touches these lines. But the parabola ω also touches these lines, and furthermore both ϵ and ω touch the polar of P for γ. Hence ϵ and ω coincide. Thus ϵ touches the principal axes of γ.

Now let γ_1 be any conic touching the four common tangents. The envelope of the polars of P for conics confocal with γ_1 is the envelope of the polars of P for conics touching these four tangents, by the

above argument, and as this is ϵ, alias ω, it follows that ω touches the principal axes of γ_1.

2. Projective generalisation of Chasles' theorem. If $t_1, \ldots t_4$ be tangents to one of a set of four-line conics touching $l_1, \ldots l_4$ and if there is a conic of the set through two opposite cuts of $t_1, \ldots t_4$ then there is one through each pair of opposite cuts, *and also* there is a conic touching $t_1, \ldots t_4$ through each pair of opposite cuts of $l_1, \ldots l_4$. Only the first part of this is effective if we reciprocate and apply to coaxal circles, to obtain theorems of Poncelet.

3. Query! What is the envelope of the principal axes of conics touching any four lines? H. G. FORDER.

1281. *A theorem on the orthopole.*

1. *The orthocentres of the four triangles formed by four lines lie on a line.*

2. *The orthopole of any one of the lines with respect to the triangle formed by the other three lies on this line.* (The orthopole of a line l with respect to a triangle ABC is found thus: let X, Y, Z be the feet of the perpendiculars from A, B, C to l, then the perpendiculars from X, Y, Z respectively to BC, CA, AB meet in a point, the *orthopole* of l for the triangle ABC.)

It is pointed out in Salmon's *Conics* (p. 246) that 1 follows by Pappus' case of Pascal's theorem. A lengthy proof of 2 is given by Clawson.* We deduce 1 and 2 together from Pappus' theorem.

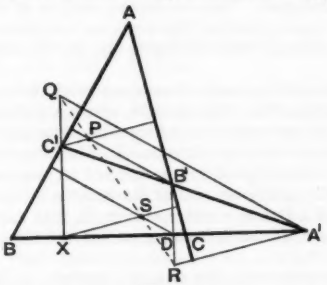


FIG.

Let AB, BC, CA, $A'B'C'$ be the four lines; let P, Q, R be the orthocentres of $AB'C'$, $BC'A'$, $CA'B'$ and let S be the orthopole of BC with respect to the triangle $AB'C'$.

First take as base-lines of Pappus' theorem, $A'B'C'$ and the line at infinity. Then the hexagon whose successive sides are $A'Q$, $B'P$, $B'R$, $C'Q$, $C'P$, $A'R$ shows that P, Q, R are collinear.

Next take as base-lines $A'BC$ and the line at infinity; then the hexagon whose successive sides are $A'Q$, DS, DR, XQ, XS, $A'R$ shows that Q, S, R are collinear.

Hence we have 1 and 2. H. G. FORDER.

* "The complete quadrilateral", *Annals of Mathematics*, 20, 1919.

REVIEWS.

Theory of the Integral. By S. SAKS. Second Edition. Pp. viii, 348. 5 U.S.A. dollars. 1937. Monografie Matematyczne, VII. (Warszawa-Lwów)

For a book of this type to achieve what is essentially a third edition within seven years is unusual; its outstanding success must be very gratifying not only to Dr. Saks and the Editorial Committee of the Monografie Matematyczne, but also to analysts everywhere in so far as it is evidence of the growing interest in our subject. In these circumstances it will be all but sufficient to describe some of the considerable changes, both in material and in arrangement, which have been made in the French edition reviewed in the *Gazette*, XIX, p. 17 and which has been, appropriately, translated by L. C. Young.

In the first chapter, we find an account of the constructive definition of an integral in an abstract space in which is given an additive system of sets and an additive (integrator) function of the sets of this system. The treatment is similar to that given in the Annexe to the 1933 edition, by means of which we obtain very general forms of the familiar theorems on integration as, for instance, those of Lebesgue and Fatou on term by term integration. Supplementing this chapter, there is a note by Banach which contains a novel, axiomatic account of the Lebesgue Integral which is entirely independent of the notion of measure.

In Chapter II, the abstract space is specialised to be a metrical space and an account of Carathéodory measure is given.

In Chapter III, the space is further specialised to be R_n, the Euclidean space of n-dimensions, and there is associated a measure, and hence an integral, with any additive function of an interval of bounded variation. This integral is the Lebesgue-Stieltjes Integral. The concluding section of this chapter contains an account of the more practical theorems on integration, such as Fubini's Theorem, the Second Mean Value Theorem and the Theorem on Integration by parts.

There is a considerable amount of new material, due mainly to the author and to Ward, in Chapter IV. This chapter, which is perhaps the most difficult, is concerned with the derivation of additive functions and in it the concepts of strong and general derivation are introduced. The ideas of strong and ordinary derivation coincide when applied to functions of an interval in R_1 but there are striking differences in spaces of higher dimensions. One of these is that an indefinite integral of a function summable in R_2 may not be derivable in the strong sense anywhere, although its integrand is its ordinary derivative almost everywhere.

Chapter V, which deals with the area of a surface, is practically unchanged from the corresponding chapter of the earlier edition.

In Chapter VI we find a careful account of the Perron Integral and its various generalisations.

Chapter VII deals with the extensions of the notions of "bounded variation" and "absolute continuity"; these are required in the next chapter which begins with the descriptive definitions of the Denjoy Integrals and concludes with their constructive definitions.

The concluding Chapter IX is concerned with the derivation of functions of one or two real variables and contains much interesting material. In particular, we notice what must surely be the natural account of the theorems of Denjoy on the distribution of the extreme derivates of functions. This treatment, due to the author and to Roger, makes use of the concept of the contingent, $\mathrm{contg}_a E$ of a subset E of R_n at a point a of E; this concept, which is due to Bouligand, is defined as follows. A half tangent to E at a is a ray l issuing

from a which is such that there is a sequence of distinct points $\{a_n\}(\neq a)$ of E, such that the directions of the rays aa_n tend to that of l; the set of all half tangents issuing from a constitutes $\mathrm{contg}_a E$. For instance, the contingents at the origin of the graphs of the functions $y=f(x)$ defined by

$$f(x)=|x|^{\alpha}\sin x^{-1}\,(x\neq 0),\ f(0)=0,$$

are the straight line $y=0$, the sectors $\left|\frac{y}{x}\right|\leqslant 1$, the whole plane R_2 in the cases $\alpha=2,\ 1,\ \frac{1}{2}$ respectively. With this definition the simplest case of Denjoy's Theorem can be expressed as follows:

If $f(x)$ is a continuous function, then for almost all values of x the contingent of the graph of $f(x)$ at the point $(x, f(x))$ is either a straight line (so that the curve has a unique tangent), or the whole plane or a half plane.

Besides the note of Banach already referred to, there is a translation of the note on Haar's Measure which he contributed to the French edition. There is also an extensive bibliography, incomplete in so far as that there is no mention of the author's personal contributions to the theory!

The general impression produced by the new edition is that it is more of a monograph and less of a textbook; on the other hand, the book is not encyclopaedic, and all will appreciate the nice judgment exercised by Dr. Saks in his choice of material and his methods of presentation of it and, in particular, in his avoidance of generalisations of uncertain importance. Dr. Saks' book will be invaluable to the advanced student and the active researcher, while the novice will find it very stimulating and he will be encouraged to continue his studies of this fundamental part of the Modern Theory of Real Functions.

Finally, the printing and paper are excellent: the only blemishes noticed being trivial misprints—doubled or undoubled consonants, or displaced symbols.

JOHN TODD.

Introduction to the Theory of Fourier Integrals. By E. C. TITCHMARSH. Pp. x, 390. 17s. 6d. net. 1937. (Oxford)

Professor Titchmarsh's new book is certain to take its place as a standard work in mathematical literature. The subject of Fourier integrals is a fairly old one, but a large number of papers have been written in recent years on Fourier integrals and on the allied subjects dealt with in this book. The subject is one which appeals to varying types of mathematicians. The theorist whose whole interest lies in formulating conditions under which general theorems are true has had ample scope here, and the formalist whose main interest lies in formulae for their own sake has had fully as much scope as the theorist. Professor Titchmarsh has shown in his many writings on the subject that he is interested in both the theoretical and the formal sides, and in this book he has catered fully for both types of mathematician.

The author states that he has not attempted to deal with a number of important topics of recent growth, such as Wiener's Tauberian theorems; applications to almost periodic functions, quasi-analytic functions, and integral functions; Stieltjes integrals; harmonic analysis in general; and Bochner's generalized integrals and the theory of functions of several variables. In view of this, no doubt the work may be regarded as an introduction to the subject, but it goes into considerable detail in the field that it professes to cover.

The first chapter, on convergence and summability, introduces Fourier's formulae, Fourier transforms, the formulae of Laplace and Mellin, and the singular integrals of Cauchy and Weierstrass. The next chapter gives the analogue of Parseval's formula, both for Fourier transforms and for Mellin transforms, and also a proof of Poisson's summation formula. Chapters III and

IV give the theory of transforms of the class L^2 and of other L classes, and these are followed by uniqueness theorems and the theories of Hilbert transforms, general transforms (including Hankel transforms) and self-reciprocal functions. Applications are then made to the evaluation of definite integrals, to the solution of differential and difference equations, such as those which occur in mathematical physics, and to the solution of integral equations, examples of which occur in pure mathematics and also in probability and statistical mechanics. The book closes with a list of books and monographs, an imposing list of original memoirs, and an index.

This book should have a wide appeal to varying types of mathematicians. Many theorems are proved under various conditions, and there are hosts of interesting formulae, such as infinite integrals involving Bessel functions and lattice-point formulae.

Professor Titchmarsh is to be congratulated on filling, in a masterly way, a gap in the literature of mathematics.

W. N. B.

An Introduction to Projective Geometry. By C. W. O'HARA and D. R. WARD. Pp. ix, 298. 12s. 6d. 1937. (Oxford)

It is, I think, generally recognized that the most satisfactory approach to the study of Projective Geometry is that which presupposes no metrical basis for the subject, but which develops the whole theory *ab initio* starting from a clearly stated set of axioms. It seems to be commonly believed, however, that such an abstract development of the subject must necessarily be so complicated and elaborate that it cannot profitably be taught to junior students, and it is an undoubted fact that most of the existing books which treat Projective Geometry from this angle are found difficult going by students who read them for the first time. Those who have attempted to teach the subject from this point of view do not share the belief, though they may often have regretted the lack of an adequate elementary textbook. The work under review, which modestly disclaims any purpose of satisfying "a long-felt want" is definitely elementary, though uncompromisingly axiomatic in its treatment, and most of it should be well within the comprehension of the freshman or scholarship candidate for whom it has been written.

Before discussing the adequacy with which the book fulfills its aim it may be useful to give a brief account of its scope. It deals exclusively with plane geometry. After a short historical introduction, it lays down the fundamental propositions of incidence and extension which are to be assumed, and having proved the consistency of the assumptions proceeds to build up the theory of projective ranges and pencils. As is well known, Desargues' theorem cannot be deduced from the propositions of incidence in a plane, though it is an immediate consequence of the corresponding propositions in space. Consequently, most treatments of the subject commence the study of plane geometry by proving a theorem of solid geometry, thereafter forgetting about the extra dimension until the time comes to develop solid geometry on its own account. Such a procedure would be out of place in a treatment like the present, in which solid geometry is not to be developed at all, and so Desargues' theorem is postulated as one of the initial assumptions—a novelty, but a perfectly logical one. After developing the theory of projective ranges and pencils (with the postulated theorem that two projective ranges with three coincident pairs of corresponding points are identical) there follows a treatment of quadrangles and quadrilaterals, of involutions, harmonic ranges, and harmonic pencils. The properties of the conic are developed from its projective definition as the locus of intersections of corresponding rays of two projective pencils, and the usual ele-

mentary properties are fully discussed, together with certain miscellaneous results such as Hesse's theorem. The question of the introduction of co-ordinates into projective geometry is discussed at length, and the last chapters deal with Euclidean and Non-Euclidean metrical geometries, showing how they may be derived from Projective Geometry by making suitable conventions.

In any treatment of the subject that is definitely introductory in scope, there is necessarily a great deal which must be omitted, and it is largely a matter of personal taste as to what should be included in addition to those parts which are absolutely fundamental. There is in any case sufficient material here for a sound course in plane Projective Geometry, and it is perhaps unreasonable to blame the authors for the omission of any specific topic not absolutely vital to the development of the subject. I feel, however, that Chapter VI, "Further Theorems on Conics", is somewhat scrappy; in its very curt treatment of pencils of conics, for instance, it is a little hard to see why alone of the special types of pencil that consisting of conics having double contact should be singled out for special mention. The whole section which deals with the conic is, in fact, by far the least satisfactory part of the book. A bad start is made at the outset, when it is asserted that, given five points U, V, A, B, C of which no four are collinear, there is a unique conic through them which can be generated by projective pencils having U and V as centres. This is not true if U, A, B lie on a line, at least in any sense in which the authors have defined projective pencils. A little further on a proof is given of the theorem that the polars of a range of points on a line with respect to a conic form a pencil projective with the range, and the proof breaks down if the line touches the conic.

These may seem to be small matters, and it is true that a small alteration of the text will put things right. But they are a lapse from the high standard of rigour which the authors have set themselves earlier in the book. For, in the initial stages of the subject, and in the introduction of a coordinate system, which are just the parts of the subject which students tend to find difficult, the authors seem to have combined clearness of exposition with rigour of treatment in a remarkable degree, without becoming too difficult for the student previously unacquainted with the subject. There are small points in these chapters that can be criticised; the phrase "any number whatever" at the foot of p. 14 is vague, especially as it subsequently appears that the "number" may be complex; and the specification of the point U at the foot of p. 49 is incomplete. Further, the appeal to the projective proposition to prove that the ranges $(ABCD)$ and $(BADC)$ are projective seems unnecessary, in view of the simple construction by which one of these ranges can be projected into the other. Apart from these minor blemishes, the chapters (II, III, IV) which deal with projectivities are excellently done, as is the introduction of coordinates in Chapter VII. The chapters on Elliptic and Hyperbolic Geometries, though outside the range of ordinary teaching nowadays, will be read with interest by the intelligent student.

The algebraic treatment of the subject is also dealt with, but less convincingly than the synthetic part. The presentation gives a curious feeling of awkwardness and general lack of elegance, without carrying any conviction that this is inherent in its nature. This may in part be due to the authors' habit of writing the steps of their algebra out in full (in itself not reprehensible in an elementary work), but whatever be the reason, the sight of cumbersome unsymmetrical equations is not one to delight the eye. The algebra on pp. 47-8 is an example in point; possibly it is unavoidable, but it is not made to appear inevitable.

In short, this book succeeds in giving a clear and simple account of those parts of the subject which are generally believed to be too difficult for im-

mature students. In demonstrating that this is possible, it has certainly fulfilled a long-felt need. It is less happy in dealing with the applications of the fundamental notions to the conic—a topic presenting far fewer pedagogic difficulties.

There are rather a lot of misprints, though few of these will cause any confusion to the reader. The diagrams are clear and numerous, and the general production has the high standard always associated with the publisher's name.

J. A. TODD.

Pseudosphärische, hyperbolisch-sphärische und elliptisch-sphärische Geometrie. By F. SCHILLING. Pp. viii, 240. RM. 12. 1937. (Teubner)

It is now nearly half a century since Klein's Lectures on Non-Euclidean Geometry were first published with the aid of Dr. F. Schilling, and there are few works that have had a greater influence on geometrical thought in the present generation. In several books, Dr. Schilling has continued to expound and elaborate the point of view and to amplify the treatment presented in the original work. His last volume, reviewed in these pages a year ago, emphasized the pseudosphere as a suitable approach to the subject, and in dealing with that surface, the usual mapping on the complex plane was used ; in the latter portion of the book it was indicated that this indirectness might have been avoided and the present volume makes a new start with the object of giving the very simplest method of attack for a lightly equipped reader. "We may try to climb a mountain peak from many different sides. And every way will present its own charm in the difficulties of the ascent. On the peak itself, however, all the different ways unite to give the same wonderful, far-reaching view. When after a time, through advancing knowledge and skill, a particularly easy path to the summit has been found, not only expert climbers but a far larger number of lightsome travellers can enjoy the beauty of the outlook from the top."

The title of the book is itself a table of contents. The geometry (and trigonometry) of the pseudosphere itself is first studied by combining three simple movements on the surface, and a geodesic mapping leads at once to Cayley's hyperbolic metric in the plane. The analytical processes adopted suggest a similar treatment of the sphere (or portion thereof) which leads to a hyperbolic or elliptic metric, according as the sphere has imaginary or real radius. Then, in the last few pages, the writer points out with much care how such a metric in the Euclidean plane is to be distinguished from a non-euclidean geometry and proceeds to give a summary account of the foundations of such a geometry by means of projective coordinates, following Klein's well-known plan. This the author has done more fully and systematically in a former work to which the reader is referred for more details.

This book is clearly intended for the young teacher and its object is to make clear to him the foundations of the geometry he is teaching by taking him through a course which works upon alternative hypotheses. Certainly this is most desirable, as there is no case in mathematics which can be stated so completely, and nothing could better the lucidity and precision of the treatment in this small volume. But, for a beginner, it must not be taken to supersede Bonola's historical summary nor indeed the original lectures of Klein, as it would fail in being at once too ample and too restricted ; too ample in that the details of analytical work, simple enough in themselves, are allowed to crowd out the main argument unless that is already in the reader's mind, and too restricted in so far as only one path to the summit is insisted upon, every point being worked out with an exactness which suggests that any attempt of the reader to proceed on his own will surely lead him over a precipice. The red-marked path of the guide book is in places artificial with its wire rope and steps.

As a matter of fact, the deductions are for the most part easy exercises, and it is unlikely that a reader will appreciate the argument unless he already has the facility in dealing with problems of projective geometry which will allow him to dispense with many of the elementary proofs. The figures throughout are extremely well drawn and marked so as to be almost self-explanatory.

P. F.

Differentialgeometrie, I : Raumkurven und Anfänge der Flächentheorie. By R. Rothe. Pp. 1-5, 6-132. RM. 1·62. 1937. Sammlung Göschen 113. (Walter de Gruyter, Berlin)

In view of the range of mathematical subjects included in the Sammlung Göschen, it is surprising that an introduction to differential geometry was not attempted long ago. This first volume deals with curves in space, with the general ideas involved in the study of a surface by means of a network of curves upon it, and with the expression of the relation between a single curve and a surface by means of the normal curvature, the geodesic torsion, and the geodesic curvature. Vectors are used freely throughout, and it is curvatures and torsions, not the more artificial radii, that are regarded as fundamental. The treatment is intrinsic but not kinematical : curvatures and torsions are introduced not as spins but as scalar multipliers of known unit vectors.

The detail most interesting, considering the scale of the book, is the early and careful mention of complex space, with a warning against the assumption that properties accepted as intuitive in real space will necessarily survive the generalization. In the section on minimal vectors, there is, however, a remark that is liable to misunderstanding : real vectors are vectors in real space, and to say that the typical nul vector is $\mathbf{a}+i\mathbf{b}$ where $\mathbf{a}$, $\mathbf{b}$ are real vectors of the same length at right angles to each other, implies that complex space can not be self-supporting but must be built on a coordinate frame belonging to real space ; this is a naive view, which loses all plausibility as soon as we begin to form any theory of analytical geometry which accounts for the axes as well as for the coordinates.

Among the subjects discussed are the determination of a curve of given curvature and torsion by the solution of a Riccati equation, and Weierstrass' formulae for minimal curves. Meusnier's theorem is shown in its double aspect, as a theorem relating the curvature of any curve on a surface to its osculating plane, and as a theorem on the curvature of a plane section. The idea of application of one surface to another is explained by a comparison in detail between the catenoid and the helicoid.

Briefly, the book is a masterpiece of compression, well maintaining the reputations both of its author and of the series to which it belongs, and we look forward to the sequel.

E. H. N.

Formelsammlung zur praktischen Mathematik. By G. Schulz. Pp. 147. RM. 1.62. 1937. Sammlung Göschen, 1110. (Walter de Gruyter, Berlin)

This short book, in the pocket-size Sammlung Göschen format, less than 150 pages of some 4 by 6 inches, is remarkable for the wealth of its contents. A rapid summary will show its range : tables of binomial coefficients, factorials and their logarithms, Bernoulli and Euler numbers and various constants ; power series for various functions ; description and use of slide rule and nomograms ; theory of errors and least squares, treated in the classical way of Gauss ; numerical solution of algebraic equations by standard methods, also of simultaneous linear equations ; interpolation, direct and inverse, numerical integration and summation ; fitting of polynomials and harmonic series by least squares ; solution of differential and partial differential equations by

standard numerical methods; a selective list of references to cognate books and memoirs.

The feat of compressing all of this within 150 small pages is attained, of course, by suppressing theoretical derivations of the formulae and methods, but the space thereby released is very well utilised in explanation and examples of procedure.

The present reviewer has only one comment to make. There is little hint in these pages of a rather revolutionary fact, which is that the increased use of the modern calculating machine, and the remarkable improvements made in it of recent years, have quite displaced the centre of gravity of practical mathematics. The standard methods are well enough if we have at the most a table of logarithms or a Crelle's table of products at our elbow; if we have a machine we shall short-circuit the methods in various ways.

In spite of this, the book before us has a basic value, and will take its place as a standard work of reference and a thesaurus of formulae and methods.

A. C. A.

Höhere Mathematik für Mathematiker, Physiker und Ingenieure. By R. Rothe. Teil IV, Hefte 5, 6. Pp. 1-54, 55-105. RM. 1.80 each. 1937. Teubners mathematische Leitfäden, 37, 38. (Teubner)

These two parts contain solutions to examples on curves and surfaces, curvilinear and multiple integrals, and differential equations, all with a strong bias towards practical applications on a sound theoretical basis. This extremely useful textbook is now complete, except that we are promised a companion volume containing a collection of formulae.

T. A. A. B.

Logical Syntax of Language. By Rudolf Carnap. Pp. xvi, 352. 25s. 1937. (Kegan Paul)

The idea of applying mathematical methods in a systematic way to the solution of philosophical problems is no novelty. It was the central principle of Russell's philosophical outlook before the War and readers of his *Introduction to Mathematical Philosophy* or *Our Knowledge of the External World* must have wondered whether his programme for the reform of philosophy has inspired useful research in the ensuing period. This elaborate treatise is the answer to the question.

To transform philosophy into a field of applied mathematics was an audacious project. That it should have seemed feasible was due to the increasing tendency for both logic and mathematics, from the middle of the nineteenth century onwards, to identify their subject matter. On the one side, the generalization of syllogistic logic and the invention of the calculus of propositions and propositional functions gave logic increasingly the appearance of an algebra of a rather peculiar sort; while from the mathematical side emancipation from interest in purely quantitative relations, of which the rapid growth of subjects like group-theory and formal algebra was an index, allowed the assimilation into mathematics proper of the newly invented calculi of logic. Finally, the definition of all the important quantitative mathematical notions in terms of purely logical notions accomplished (or nearly accomplished) in *Principia Mathematica*, made the boundary between the two subjects appear increasingly a matter for arbitrary decision.

From this period of development the trivial syllogistic logic of the scholastics emerged as a powerful mathematical discipline, fit to marshall and order formal relations in any field of abstract inquiry. Now logical analysis, the symbolising of formal relations in a field by means of a calculus amenable to mathematical manipulation, was the only function which Russell's destructive criticism of speculative philosophy left for the philosopher to do. Thus philo-

sophy, or the subject which was to take its place, became identical with logic, a branch of applied mathematics.

The twenty-five years or so which have passed since Russell's programme was formulated have produced a wealth of interesting researches. In the purely technical aspects of symbolic logic, the invention of systems of " strict implication " by C. I. Lewis and others, and of multi-valued logics (Tarski, Lukasiewicz), the researches into meta-mathematics by Hilbert's school, and Gödel's remarkable investigations into the limitations of mathematical symbolism, have revealed the defects of earlier conceptions of formal logic while providing the materials for their remedy. Still more striking than this advance in technique is the fact that Russell's general philosophical background has largely inspired the creation of an active philosophical movement, now known as Logical Positivism, of which Professor Carnap is one of the foremost exponents.

The aims of this movement, which are very clearly explained in the last section of this book (" Philosophy and Syntax ") are to replace " the inextricable tangle of problems known as philosophy " by the logical analysis of scientific findings. Thus the new philosophical method is not itself empirical, but applies mathematical methods to clarify and order the theories and data of empirical science.

In the earlier sections of the book, Professor Carnap shows the method in action by applying it to the analysis of two artificial " languages " or symbol-systems. The first is constructed to satisfy the demands of Brouwer and the intuitionists, the second, which is more elaborate, contains symbolic apparatus adequate for the representation of any mathematical or physical theory, however complex. By making the languages themselves the subject of his investigations, Professor Carnap is able to define all the logical notions needed in the investigation of the structure of a science. An important feature of the method is its *formal* character, attention being paid always to kinds and arrangements of symbols, never to their meaning. This allows the translation of structural properties of theories into arithmetical properties of certain integers by means of an ingenious method due to Gödel. These formal investigations, some of them a lucid paraphrase of the work of others, but many original, will have permanent value even for those who do not share Professor Carnap's philosophical background. The demonstration of the far-reaching independence of logical analysis of questions of meaning, and the numerous careful and illuminating discussions of such questions as the mathematical antinomies, the theory of types, the possibility of impredicative definitions, the use of languages to make assertions concerning themselves, and many more, deserve the serious attention of the specialist.

For those who are not experts in the field of symbolic logic, the book will, in spite of masterly clarity in presentation, undoubtedly prove difficult reading. Such might be well advised to read first the same author's little book on *Philosophy and Logical Syntax* (Kegan Paul, 1935) which makes an excellent introduction to the larger work. In the latter, mathematicians interested in the philosophy of their subject or merely curious to discover a somewhat unlikely use for this subject, will find much fascinating material, even in a casual reading of isolated sections. For the specialist, this book will take its place on his bookshelves next to the *Principia*.

The translation (by the Countess von Zeppelin) is very good, smooth, easy to read, and faithful to the original ; the publishers must be congratulated on the handsome appearance of the book—the complex symbolism (ranging from Clarendon to Gothic type) is very legible and the text is remarkably free from misprints.

M. BLACK.

The Handmaiden of the Sciences. By E. T. BELL. Pp. viii, 216. 9s. 1937. (Baillière, Tindall and Cox)

This book aims at describing, for the general reader, the service which mathematics has rendered the sciences. One would know better how to review it if one knew who would read it; and we are all aware that the world is obtuse to the successes of mathematics and ignorant of its nature. So that when the book falls into the hands of the right reader, he will find in it much to stimulate him and much good information on the aims of mathematics. It was a good idea, for example, to put in boundary conditions and an introduction to invariants. And there are many passages in which the author writes clearly and boldly and lays out the nature of mathematics in a way that can hardly fail to grip the semi-scientific reader, or the ordinary reader of intelligence. The author is well aware of the futility of some methods of teaching, and of the uncertainty and lack of exactness of mathematical reasoning; well aware also of the feelings of the researcher. In some ways the book will leave too much the impression of a person who has approached the modern physical theories from the mathematical point of view. There is too much logic, too much stress on the logical basis of mathematics and too much on those parts of mathematics which depend on logical assumptions; too much also about the Greeks and their way of thinking. The author may have begun as a pure mathematician and approached physical theory through relativity or the quantum theory; both, in their present form, logical structures. There is a certain diffuseness of aim, which contrasts with the objectivity, over a smaller range, of Whitehead, in his *Introduction to Mathematics*. The book consists in part of general principles and their elucidation and in part of explanations of the elements of mathematical analysis, such as the nature of differentiation, total and partial, after the manner of Whitehead. But what strikes us forcibly is the lack of uniformity of plan. Sometimes we have passages of real eloquence, followed at once by atrocious jokes and colloquialisms; and the worst of it is that the jokes are not always appropriate but apt to be dragged in after the manner of an after-dinner speaker or newspaper hack. Thus, in describing damped vibrations, he writes:

"The reader is no doubt familiar with one perfect instance of such vibrations in the Mouse's Tale as recorded in *Alice in Wonderland*, where the printed Tale is presented in the sinuous curve of the mouse's tail. Lewis Carroll perpetrated one pun intentionally and possibly two, for the vibrations of the tail would necessarily be somewhat damp after the mouse had swum the pool of tears."

The present state of the theory of the ether is summed up as follows:

"As they say in the barber shops, 'Who's next?' In the dim light of current physical speculation the head barber bears a ghostly likeness to William of Occam. But whether or not any of the bewhiskered theories reading the *Saturday Evening Post* or the latest tabloids of relativity and the quantum theory as they wait their turn are to be shaved in our generation, one thing seems fairly certain: Maxwell's equations will not be swept up by the whiskers of the bootblack."

Lastly, we have gratuitous comments like the following:

"Limitations of space forbid more than a passing mention of the remarkable prophecy—'hunch' is perhaps juster—of the great algebraist, J. J. Sylvester."

An English judge of the traditional type would ask "What is a hunch?"; and indeed we had to ask before we could fathom this passage. This style of writing will not improve the reputation of the author.

Another feature which is sometimes hard to understand, and sometimes merely annoying, is an all-pervading reference to certain villains: the villains

are not mentioned, but their activities are only hinted at, so that the present reviewer, for one, is in the dark in many instances. Thus:

"When this theory [the quantum theory] first burst on a disconcerted science, it was accompanied by the most agonizing appeals to a non-existent physical intuition that physics had ever known."

The villains can tremble, for I do not know who they are.

A book of this kind must contain many opinions that all cannot agree with. That there are a few errors is not surprising. We congratulate the proof-reader on getting Maupertius in more often than Maupertuis, a phenomenon which all who have had to deal with that arrogant race will appreciate. On the whole we should bless the book more if it bore fewer signs of facile writing.

F. B. P.

A Course in Geometry. By J. L. LATIMER AND T. SMITH. Pp. 388. Complete 4s. 6d.; also in two parts, 2s. 6d. each. 1937. (Harrap)

This course is a sensible one; its primary purpose is to exploit the sensible idea that single letters should be used for angles, not only in riders but in writing out the standard theorems. This necessitates some drill to "ensure large clear figures in which the angles are plainly and legibly marked by single letters", but this is all to the good and the authors claim that it leaves the pupil "free to concentrate on the purely mathematical difficulties of the subject".

There is a preliminary or Stage A course, consisting entirely of examples. The introduction of angles in degrees seems too hurried; there is no chance to learn what a right angle is; [corners of tables, etc.; vertical and horizontal; about turn, right turn, etc.] Angle is made to depend only on the clock and the protractor and Ex. 2 on angles is: "Draw any triangle. Measure the three angles of the triangle. What is their sum?" There is in this part little help to the teacher in devising an ordered and progressive course. There is, however, a nice set of three-dimensional problems, and also of locus problems.

The deductive course is divided into "the usual first, second, third and fourth year's work in secondary schools", each of these ending with a group of the standard constructions and a set of revision papers.

The first year's work contains: Part I, Fundamental Facts, and Part II, Geometrical Theorems. In the former, the useful phrase "the pupil should satisfy himself that he agrees with the following assumptions" followed by the various needed facts set out as Assumption 1, Assumption 2 and so on, keeps the authors well away from the vicious circle, in which so many involve themselves, in which parallel lines are defined as those in the same direction and the same direction at different points as that of parallel lines. The theorems in Part II are those of Angle-sum, Congruence, Isosceles triangle, Inequalities, Parallelograms, Intercepts. It is perhaps interesting to note, as an illustration of the difficulty placed on authors since the pupil has been relieved of having to learn proofs of the congruence theorems, that Theorem I (Angle-sum) is introduced thus, "We now come to a new group of geometrical facts, the theorems, each of which requires systematic proof" but with Theorem II (Euclid I, 4) we read "a 'proof' is given on p. 370". A curious point, but very possibly sound, in the order is that the angle-sum for a quadrilateral follows that for a triangle but the angle-sums for a polygon's interior and exterior angles are reserved for Theorems 20 and 21. The exercises in this part and throughout the book are usually divided into Section A for the average pupil's first reading and Section B for revision and the better pupils. This is an attractive feature.

The second year's work is Part III, Areas; this is very short, but presumably

considerable revision would be needed. It includes areas of parallelograms and triangles; Pythagoras' theorem; constructions and an unusually vigorous attack on loci. The authors, in dealing with the two usual locus theorems, are not satisfied with two parts to each but give *three*, using the word *locus* in the third only.

The third year's work is Part IV, the Circle. In the "alternate segment" theorem (44) an awkward point for the single-letter notation occurs. We are told to "prove that $a=b$ and $x=y$" but neither a nor x appears in the figure. Since each of these angles is cut in two, we have to be told to denote $\angle DBC$ by a and $\angle DBA$ by x, so the proof could be shortened by using neither a nor x; however, it probably looks better with them. This point also arises in one case of Theorem 33.

The fourth year's work is Part V, Further Areas and Ratio. Here we have geometrical illustration of algebraic identity, extensions of Pythagoras, Apollonius, and then the usual Ratio and Similarity theorems ending with Theorem 61, Areas of Similar Triangles, and including Ptolemy's theorem set out in full but not numbered.

The "very short course" beyond School Certificate standard whoch follows consists of ten pages including Ceva, Menelaus, Simson line, Nine-point Circle.

The exercises throughout seem adequate and are made more attractive by the use of a fair number of small inset figures. The pagination is good as regards the tops of the pages, but less so lower down as many pages are half empty.

In the deductive course, the explanations are clear and well arranged.

There is not much care shown to secure consistency in details even with regard to the distressing and almost universal use of degrees. Thus: Theorem 7. Two right-angled triangles have $B=Y=90°$; Construction 5. $x+m=180°$ (half a revolution), $\therefore x=90°$; Theorem 24. $\angle ABC=1$ right angle, $\angle XCB=\angle ACL$ (each $=90°+x$).

Again the symbol $<$ is explained to mean "is less than" and in the next line is used for "less than".

As a rule, the heading "Theorem 7", or whatever may be the number, is followed by the enunciation in black type, but the enunciation of Theorem 41 is "Theorems connecting arcs, chords and angles in equal circles are comprised under this heading" followed by four theorems with no attempt at proof.

As regards the "tangencies" a curious course is followed. The special case of point, line, circle (P.L.C.) when the point is on the line is written out in full and so is P.P.C. later, but P.P.L. and P.L.L. appear only as exercises and P.P.L. only in a numerical case.

However, these last criticisms are small ones and do not alter the fact that, as a whole, the deductive course makes an excellent impression.

C. O. Tuckey.

Hydro- and Aero-dynamics. By S. L. Green. Pp. viii, 166. 12s. 6d. 1937. (Pitman)

In this small volume, the author introduces his readers to all the more important lines of attack upon the problems within the scope of his title. After a chapter upon the fundamental equations, there follow two upon irrotational motion in two dimensions, including discontinuous motion with free streamlines, and giving the relevant portions of the theory of functions of a complex variable. The next two chapters are mainly devoted to general theorems, such as the circulation theorem and the Blasius contour integrals for the force and torque on a cylinder in a stream, although it is here that the Joukowski transformation of a circle into an aerofoil section is to be found. Then follow

chapters upon vortex motion in two dimensions, three-dimensional motion, and viscous fluids, the last of these containing a well-chosen set of representative problems. In the final chapter on flow at high Reynolds numbers the reader is introduced to Prandtl's boundary layer equations, and a brief mention of turbulence ends the book. A large number of exercises is provided, and these are largely drawn from examination papers of the University of London.

To have accomplished so much in the compass must have meant a rigid selection and rejection of topics, but it is difficult to suggest any that might with advantage replace those included. At the same time, comprehensiveness has been achieved too often by stating solutions and verifying them, rather than by building them up from the data. It may very well happen that students will feel bewildered by this almost magical production of the correct formula, and will fail to see the subject as more than a series of "lucky flukes". Not until the chapter on viscosity does it often appear that what we are doing is really solving a set of partial differential equations subject to prescribed boundary conditions. In addition—again presumably due to limits of space—the author too seldom discusses the solutions he has found, and the failure to do this may make it difficult for the student to realise and appreciate the physical implications of the mathematics. As an instance, in dealing with the very important and fundamental case of the flow past a cylinder about which there is also circulation, the kinematics, and the position of the stagnation points, are well treated in one paragraph, and in the next the resultant force on the cylinder is calculated, but it is left to the reader's imagination to see that there is a physical connection between the displacement of the stagnation points (points of maximum pressure) and the existence of the thrust perpendicular to the stream.

One use of notation is perhaps unhappy. In most of the book the suffix notation is used for partial derivatives, but in the chapter on viscous fluids the author, presumably recognising the undesirability of using u_{xx} as a partial derivative and p_{xx} as a stress component on the same page, reverts to the ∂ notation.

There are, however, many points in the detailed treatment that deserve commendation, and as salient instances we may quote the introduction of "spin" or "vorticity" in terms of the angular momentum of a small fluid element (i.e. in mechanical, and not in merely kinematical, terms), and the discussion of the irrotational motion about a rotational core which precedes the mathematical abstraction of the line vortex.

W. G. B.

Further Mechanics and Hydrostatics. By A. W. SIDDONS, K. S. SNELL and N. R. C. DOCKERAY. Pp. viii, 184. 3s. 6d. 1937. (Arnold)

This book has been produced by the authors to follow their *Elementary Mechanics* (see *Gazette*, Oct. 1936) and to cover the further ground required for the Mechanical Sciences Qualifying, the Army Entrance (Higher Mathematics papers), and Higher Certificate (ordinary papers) Examinations. It covers a good deal of ground, including the Link Polygon, Variable Forces, Rotation about a Fixed Axis, and three chapters on Hydrostatics, as well as the usual subjects dealt with in similar books; though, of course, it will only provide a first reading for the mathematical specialist. The Hydrostatics will be found useful by boys taking Applied Mathematics papers which include a few questions on that subject.

The authors have paid special attention, by means of "Hints for Examples", etc., to helping the pupil solve examples, and the teacher using the book will need to ensure that the importance of the principles is not minimised. An

attractive feature is the way in which Calculus methods and ideas are made use of wherever suitable. At the end is a set of " Bookwork Questions " and an index. J. W. H.

Manual of Mathematics and Mechanics. By G. R. CLEMENTS and L. W. WILSON. Pp. vii, 266. 15s. 1937. (McGraw-Hill)

We have not often seen a mathematical book so beautifully produced as this one ; the paper is a pleasure to touch and the pages easy to turn ; the printing of the tables, the formulae and the graphs of standard functions is excellent. The contents cover a wide range and include : tables of logarithms, trigonometrical and hyperbolic functions and their inverses, square and cube roots and reciprocals ; thirty pages of standard integrals ; formulae used in calculus, trigonometry, coordinate geometry and mechanics ; tables of moments of inertia, centroids and bending moments.

Although the book is so comprehensive we cannot help feeling that anyone who is likely to use it will neither want tables of logarithms and trigonometrical functions without mean differences nor will he require the formulae used at the beginning of mensuration, indices, logarithms, ratios, etc. Some of the room given to the most elementary formulae in each section might well have been used for other things. Considering the large number of pages devoted to standard integrals, a few lines might have been spared for Fourier series for simple functions, such as x^2 and $\sin kx$, and also for the particular integrals of linear differential equations with rules for the use of operators. In both these examples only the formulae for the general case are stated. H. V. L.

Examples in Practical Mathematics for first-year students. By T. H. Fallows. Pp. vii, 71. 1s. 3d. 1937. (Dent)

These examples are chosen for classes in evening institutes and junior technical schools which prepare students for the S. 1 year of the ordinary national certificate course. Roughly there are fifteen pages devoted to arithmetic, eight to graphs of statistics, sixteen to geometry and mensuration, and twenty-one to algebra. The examples are very well chosen, but there is not enough drill in elementary arithmetic or in substituting numerical values in algebraic formulae, and the later examples in algebra and mensuration are on the difficult side.

There is a growing opinion that, in the whole course up to the national certificate, we attempt too much. We probably ought to expect students who have done a preparatory course to have only a thorough grounding in arithmetic, plotting graphs from tables and the expression of a sentence by means of a formula ; then they can start off in S. 1 with the manipulation of algebraic expressions, without having to do five or six weeks revision of arithmetic as they often have to do at present.

The book is very clearly printed, it includes tables of logarithms, but does not contain answers. H. V. L.

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